On the Effective Description of Multiple M2-Branes

Christian Sämann

School of Mathematics, Trinity College Dublin

University of Amsterdam, March, 24th 2009

Based on:

- S. A. Cherkis, CS, Phys. Rev. D78 (2008) 066019, [0807.0808]
- S. A. Cherkis, V. Dotsenko, CS, 0812.3127
- C. I. Lazaroiu, D. McNamee, CS and A. Zejak, 0901.3905

Outline

- Review part
  - The Nahm equation or D1-D3 branes
  - The Basu-Harvey equation or M2-M5 branes
  - Stacks of flat M2-branes: The BLG model
- Superspace formulations of BLG-like models
  - Manifestly $\mathcal{N} = 2$ supersymmetric formulation
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- Generalized 3-Lie algebras and BLG-like models
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  - Representations on $*$-algebras
- The framework of strong homotopy Lie algebras
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  - The SYM equation as hMC equations
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The Nahm Equation or D1-D3-Branes

In type IIB string theory, monopoles can be seen as D1-branes ending on D3-branes.

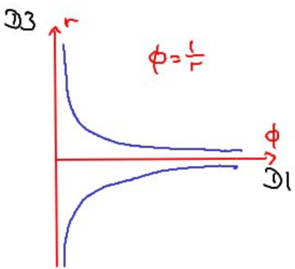
Consider a D3-brane in directions 0234.

A BPS solution to the SYM equations is the magnetic monopole with Higgs field $\phi \sim \frac{1}{r}$: A D1-brane appears.

As they are BPS, one trivially forms a stack of N D1-branes.

From the perspective of the D1-brane, the effective dynamics is described by the Nahm equations:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^i + \varepsilon^{ijk}[X^j, X^k] = 0 \ .$$



dim	0	1	2	3	4
$D1$	×	×			
$D3$	×		×	×	×

These equations have the following solution ("fuzzy funnel")

$$X^i = r(\phi)G^i \ , \quad r(\phi) = \frac{1}{\phi} \ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

The Nahm Equation and The Fuzzy Funnel

In type IIB string theory, monopoles can be seen as D1-branes ending on D3-branes.

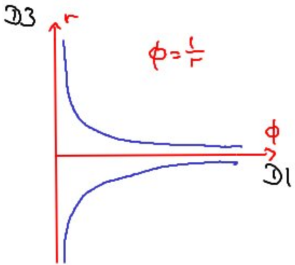
$$X^i = r(\phi) G^i \ , \quad r(\phi) = \frac{1}{\phi} \ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

Interpretation of this solution:

The $N \times N$-matrices G^i form a representation of $SU(2)$ and satisfy $\mathrm{tr}\,(G^i G^i) \sim N$, thus they are coordinates on a fuzzy S^2.

At every point ϕ, the cross section of the D1s' worldvolume is a fuzzy sphere with radius $r(\phi)$. In the limit $N \to \infty$, a smooth S^2 appears.

dofs: $R \sim N$, dofs $\sim R^2 \sim N^2$ ✓



dim	0	1	2	3	4
$D1$	×	×			
$D3$	×		×	×	×

The Basu-Harvey Equation or M2-M5-Branes

M2 branes ending on M5 branes should be described by Nahm-type equations.

M5-brane in directions 013456:

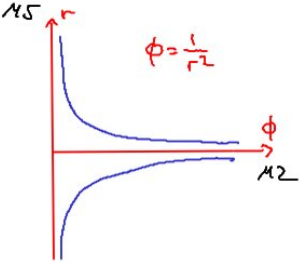
$$G^{mn}\nabla_m\nabla_n X^{a'} = 0$$
$$G^{mn}\nabla_m H_{npq} = 0$$

Ansatz for a soliton:

$$X^{5'} = \phi$$
$$H_{01m} = v_m \quad H_{mnp} = \varepsilon_{mnpq}v^q$$

Solution:

$$H_{01m} \sim \partial_m\phi \quad \phi \sim \frac{1}{r^2}$$



dim	0	1	2	3	4	5	6
$M2$	×	×	×				
$M5$	×	×		×	×	×	×

Perspective of M2: postulate four scalar fields X^i, satisfying

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^i + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0$$

Basu, Harvey, hep-th/0412310

Basu-Harvey equation:

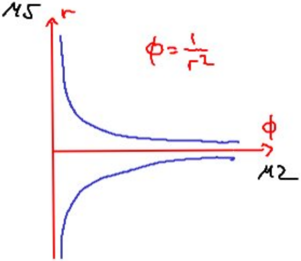
$$\frac{\mathrm{d}}{\mathrm{d}\phi} X^i + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0$$

Solution (similar to D1-D3 case):

$$X^i = r(\phi) G^i \quad r(\phi) = \frac{1}{\sqrt{\phi}}$$
$$G^i = \varepsilon^{ijkl}[G^j, G^k, G^l]$$

Interprete this again as a fuzzy funnel, this time with a fuzzy S^3 at every point ϕ (not quite...).

$R \sim \sqrt{N}$ dofs $\sim R^3 \sim N^{3/2}$ ✓



dim	0	1	2	3	4	5	6
$M2$	×	×	×				
$M5$	×	×		×	×	×	×

Metric 3-Lie Algebras

3-Lie algebras come with a triple bracket and an induced Lie algebra structure.

metric 3-Lie algebras (Filippov, 1985)

$\mathcal{A}$ a real vector space with a bracket $[\cdot,\cdot,\cdot] : \Lambda^3\mathcal{A} \to \mathcal{A}$ satisfying

$$[A,B,[C,D,E]] = [[A,B,C],D,E] + [C,[A,B,D],E] + [C,D,[A,B,E]] \quad \text{(FI)}$$

and a bilinear symmetric map $(\cdot,\cdot)_\mathcal{A} : \mathcal{A}\otimes\mathcal{A} \to \mathbb{R}$ satisfying

$$([A,B,C],D)_\mathcal{A} + (C,[A,B,D])_\mathcal{A} = 0 \quad \text{(Cmp)}$$

There is a map from $\mathcal{A}\wedge\mathcal{A}$ to $\mathrm{Der}(\mathcal{A})$ given by linearly extending

$$D_{A\wedge B}(C) := [A,B,C]\ , \quad A,B,C \in \mathcal{A}$$

The inner derivations $\mathfrak{g}_\mathcal{A} := \mathrm{im}(D_{\mathcal{A}\wedge\mathcal{A}})$ form a Lie algebra.
Two invariant pairings on $\mathfrak{g}_\mathcal{A}$: $(A\wedge B, C\wedge D)_\mathfrak{g} := ([A,B,C],D)_\mathcal{A}$
and induced Killing form.

The Metric 3-Lie Algebra A_4

The 3-Lie algebra A_4 is the most important 3-Lie algebra in the context of BLG.

Consider the vector space $A_4 := \mathbb{R}^4$ with basis $\tau_1, \dots, \tau_4$. Then define the bracket $[\cdot,\cdot,\cdot] : \Lambda^3 A_4 \to A_4$ as the linear extension of

$$[\tau_a, \tau_b, \tau_c] = \sum_d \varepsilon_{abcd} \tau_d \quad .$$

Also, the bilinear symmetric map $(\cdot,\cdot)_{A_4} : A_4 \otimes A_4 \to \mathbb{R}$ is given as the linear extension of

$$(\tau_a, \tau_b)_{A_4} = \delta_{ab} \quad .$$

Ass. Lie alg. $\mathfrak{g}_{A_4} := \mathrm{im}(D_{A_4 \wedge A_4}) = A_4 \wedge A_4$ generated by $\tau_a \wedge \tau_b$, which satisfy the commutator relations for $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$.

The bilinear symmetric map on $\mathfrak{g}_{A_4}$ has nonvanishing entries:

$$(\tau_1 \wedge \tau_2, \tau_3 \wedge \tau_4) = (\tau_1 \wedge \tau_3, \tau_4 \wedge \tau_2) = (\tau_1 \wedge \tau_4, \tau_2 \wedge \tau_3) = 1$$

Approaching the Effective Description of M2-Branes

Spacetime symmetries and BPS equations give helpful constraints on the description.

A stack of flat M2-branes in $\mathbb{R}^{1,10}$ should be effectively described by a conformal field theory with the following constraints:

Spacetime symmetries: $\mathrm{SO}(1,10) \to \mathrm{SO}(1,2) \times \mathrm{SO}(8)$
extended by $\mathcal{N} = 8$ SUSY.

Field content: X^I, $I = 1, ..., 8$, and superpartners Ψ_α

Assumption

Take BPS/SUSY transformations from Basu-Harvey equation and therefore the matter fields take values in a metric 3-Lie algebra.

$$\delta X = \mathrm{i}\Gamma_I \bar{\varepsilon}\Gamma^I \Psi \qquad \delta\Psi = \partial_\mu X \Gamma^\mu \varepsilon - \tfrac{1}{6}[X, X, X]\varepsilon$$

Recipe: Try to close SUSY algebra. Constraints yield equations of motion for matter fields.

The Bagger-Lambert-Gustavsson Model

This model is an unconventional supersymmetric Chern-Simons matter theory.

BLG found that for SUSY, we need to introduce gauge symmetry.
$\Rightarrow$ Field content: $X \in \mathcal{A}$, $\Psi \in \mathcal{A}$ and gauge potential $A_\mu \in \mathfrak{g}_\mathcal{A}$.

Simplify: Clifford alg. $Cl(\mathbb{R}^{1,10})$, $X := \Gamma_I X^I$, $\{\Gamma_I, \Gamma_J\} = 2\eta_{IJ}$
$(A,B)_{\mathcal{A}\otimes Cl} := \frac{1}{32}\,\mathrm{tr}_{Cl}\big((A,B)_\mathcal{A}\big)$, $[\cdot,\cdot,\cdot]$ linearly ext.

The Bagger-Lambert-Gustavsson model

$$\begin{aligned}\mathcal{L}_{\mathrm{BLG}} \;=\; &+\tfrac{1}{2}\varepsilon^{\mu\nu\kappa}\left((A_\mu, \partial_\nu A_\kappa)_\mathfrak{g} + \tfrac{1}{3}(A_\mu, [A_\nu, A_\kappa])_\mathfrak{g}\right)\\ &- \tfrac{1}{2}(\nabla_\mu X, \nabla^\mu X)_{\mathcal{A}\otimes Cl} + \tfrac{\mathrm{i}}{2}(\bar{\Psi}, \Gamma^\mu \nabla_\mu \Psi)_\mathcal{A}\\ &+ \tfrac{\mathrm{i}}{4}(\bar{\Psi}, [X, X, \Psi])_\mathcal{A} - \tfrac{1}{12}([X,X,X],[X,X,X])_{\mathcal{A}\otimes Cl}\end{aligned}$$

This model is invariant under the supersymmetry transformations:

$$\delta X = \mathrm{i}\Gamma_I \bar{\varepsilon}\Gamma^I \Psi\,, \qquad \delta\Psi = \nabla_\mu X \Gamma^\mu \varepsilon - \tfrac{1}{6}[X,X,X]\varepsilon\,,$$
$$\delta A_\mu = \mathrm{i}\bar{\varepsilon}\Gamma_\mu (X \wedge \Psi)$$

Consistency checks

The BLG model passes a number of consistency checks.

$$\begin{aligned}\mathcal{L}_{\rm BLG} \;=\; &+\tfrac{1}{2}\varepsilon^{\mu\nu\kappa}\left((A_\mu,\partial_\nu A_\kappa)_{\mathfrak{g}}+\tfrac{1}{3}(A_\mu,[A_\nu,A_\kappa])_{\mathfrak{g}}\right)\\ &-\tfrac{1}{2}(\nabla_\mu X,\nabla^\mu X)_{\mathcal{A}\otimes Cl}+\tfrac{\mathrm{i}}{2}(\bar\Psi,\Gamma^\mu\nabla_\mu\Psi)_{\mathcal{A}}\\ &+\tfrac{\mathrm{i}}{4}(\bar\Psi,[X,X,\Psi])_{\mathcal{A}}-\tfrac{1}{12}([X,X,X],[X,X,X])_{\mathcal{A}\otimes Cl}\end{aligned}$$

Further results:

- The model is classically conformal and seems rather unique.
- The model is parity invariant.
- Under some assumptions: reduction mechanism M2$\longrightarrow$D2.

(Mukhi, Papageorgakis,0803.3218)

- Recast into the ABJM version, it yields integrable spin chain.

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Manifestly $\mathcal{N} = 2$ SUSY Formulation

There is a manifestly $\mathcal{N} = 2$ SUSY formulation, allowing for various deformations.

Approach: Take $\mathcal{N} = 1$, 4d superspace $\mathbb{R}^{1,3|4}$ and reduce to 3d.

Field content of the theory:

- The matter fields X^I, Ψ are encoded in four chiral multiplets:

$$\Phi^i(y) = \phi^i(y) + \sqrt{2}\theta\psi^i(y) + \theta^2 F^i(y) \ ,$$

- The gauge potential A_μ is contained in a vector superfield:

$$\begin{aligned} V(x) \;=\; & -\theta^\alpha\bar{\theta}^{\dot{\alpha}}(\sigma^\mu_{\alpha\dot{\alpha}}A_\mu(x) + \mathrm{i}\varepsilon_{\alpha\dot{\alpha}}\sigma(x)) \\ & + \mathrm{i}\theta^2(\bar{\theta}\bar{\lambda}(x)) - \mathrm{i}\bar{\theta}^2(\theta\lambda(x)) + \tfrac{1}{2}\theta^2\bar{\theta}^2 D(x) \ , \end{aligned}$$

$\mathcal{N} = 2$ superspace formulation of BLG (Cherkis, CS, 0807.0808)

$$\begin{aligned} \mathcal{L} = \int \mathrm{d}^4\theta \ \kappa \left(\mathrm{i}(V, (\bar{D}_\alpha D^\alpha V))_{\mathfrak{g}} + \tfrac{2}{3}(V, \{(\bar{D}^\alpha V), (D_\alpha V)\})_{\mathfrak{g}}\right) \\ + (\bar{\Phi}_i, \mathrm{e}^{2iV} \cdot \Phi^i)_{\mathcal{A}} + \alpha\left(\int \mathrm{d}^2\theta \ \varepsilon_{ijkl}([\Phi^i, \Phi^j, \Phi^k], \Phi^l)_{\mathcal{A}} + c.c.\right) \end{aligned}$$

Manifestly $\mathcal{N}=2$ SUSY Formulation

There is a manifestly $\mathcal{N}=2$ SUSY formulation, allowing for various deformations.

$$\mathcal{L} = \int \mathrm{d}^4\theta\ \kappa \left(\mathrm{i}(V,(\bar{D}_\alpha D^\alpha V))_{\mathfrak{g}} + \tfrac{2}{3}(V,\{(\bar{D}^\alpha V),(D_\alpha V)\})_{\mathfrak{g}}\right)$$
$$+ (\bar{\Phi}_i, \mathrm{e}^{2iV}\cdot\Phi^i)_{\mathcal{A}} + \alpha\Big(\int \mathrm{d}^2\theta\ \varepsilon_{ijkl}([\Phi^i,\Phi^j,\Phi^k],\Phi^l)_{\mathcal{A}} + c.c.\Big)$$

Observations:

- Superfield description of BLG analogous to that of SYM.
- This Lagrangian is not manifestly gauge invariant.
- There are various $\mathcal{N}=2$ deformations.
- Deforming by a Yang-Mills term breaks conformal invariance, but might lead to new dualities.

Manifestly $\mathcal{N}=4$ Supersymmetric Formulation

Projective superspace provides a way of making manifest $\mathcal{N}=4$ SUSY in 3d.

Projective superspace in 4d

$\mathcal{N}=2$ SUSY covariant derivatives on $\mathbb{R}^{1,3|8}$:

$$\{D_{i\alpha},D_{j\beta}\} \;=\; 0 \quad \{\bar{D}^i_{\dot\alpha},\bar{D}^j_{\dot\beta}\} \;=\; 0 \quad \{D_{i\alpha},\bar{D}^j_{\dot\alpha}\} \;=\; -2\mathrm{i}\delta_i^j\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

add $\zeta\in U_0\subset\mathbb{C}P^1$ parameterizing $\mathcal{N}=1$ within $\mathcal{N}=2$:

$$\nabla_\zeta = D_1+\zeta D_2\ , \quad \bar\nabla_\zeta = -\zeta\bar{D}^1+\bar{D}^2$$

Projective superspace: $\mathbb{R}^{1,3|8}\times\mathbb{C}P^1$ "divided by" ∇_ζ, $\bar\nabla_\zeta$.

Perform again a dimensional reduction:

$$\mathbb{R}^{1,3|8}\times\mathbb{C}P^1\to\mathbb{R}^{1,2|8}\times\mathbb{C}P^1.$$

Field content of the BLG model:

- Matter X^I, Ψ: $\mathcal{N}=1$ 4 chiral multiplt., $\mathcal{N}=2$ 2 hypermultiplt.

$$\eta_k \;=\; \bar\Phi\frac{1}{\zeta^2}+\bar\Sigma\frac{1}{\zeta}+X-\zeta\Sigma+\zeta^2\Phi$$

- Gauge A_μ: $\mathcal{N}=1$ vector multiplet, $\mathcal{N}=2$ tropical multiplet

$$\mathcal{V}(\zeta,\bar\zeta) \;=\; \sum_{n=-\infty}^{\infty}v_n\zeta^n$$

Manifestly $\mathcal{N} = 4$ Supersymmetric Formulation

In projective superspace, one can make $\mathcal{N} = 4$ SUSY in the BLG model manifest.

Field content: tropical multiplet $\mathcal{V}$ and hypermultiplets η_k.

Supersymmetric action: (Cherkis, Dotsenko, CS, 0812.3127)

$$\int \mu\, \kappa\Big(\mathrm{i}(\mathcal{V}, (\bar{\mathcal{D}}_\alpha \mathcal{D}^\alpha \mathcal{V}))_{\mathfrak{g}} + \tfrac{2}{3}(\mathcal{V}, \{(\bar{\mathcal{D}}^\alpha \mathcal{V}), (\mathcal{D}_\alpha \mathcal{V})\})_{\mathfrak{g}}\Big) + \big(\bar{\eta}_k, \mathrm{e}^{2\mathrm{i}\mathcal{V}} \cdot \eta_k\big)_{\mathcal{A}}$$

Observations:

- Chern-Simons term completely reduces to $\mathcal{N} = 1$ form.
- The complex linear superfield Σ in the hypermultiplet
$$\eta_k = \bar{\Phi}\frac{1}{\zeta^2} + \bar{\Sigma}\frac{1}{\zeta} + X - \zeta\Sigma + \zeta^2\Phi$$
can be dualized to a chiral multiplet.
- To compute the interaction terms, one would have to solve a Riemann-Hilbert problem. However, its symmetries tell us that this is the BLG model.

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Extending The Structure of A 3-Lie Algebra

The notion of a 3-Lie algebra is too restrictive and one has to find a generalized notion.

Problem: Given a three-algebra $\mathcal{A}$, if its bilinear form $(\cdot,\cdot)_\mathcal{A}$ is positive definite, then $\mathcal{A}$ is A_4 or a direct sum thereof.

A_4 supposedly describes a stack of 2 M2-branes, not enough.

Mukhi, Papageorgakis, 0803.3218

Possible extensions:

(1) Assume, 3-Lie algebras appear accidentally $\Rightarrow$ ABJM model
(2) Give up positive definiteness of $(\cdot,\cdot)_\mathcal{A}$ $\Rightarrow$ ghosts
(3) Relax conditions on 3-Lie algebras

Guideline: Demand gauge invariance of the $\mathcal{N}=2$ Lagrangian

$$\mathcal{L} = \int \mathrm{d}^4\theta\ \kappa\left(\mathrm{i}(V,(\bar{D}_\alpha D^\alpha V))_\mathfrak{g} + \tfrac{2}{3}(V,\{(\bar{D}^\alpha V),(D_\alpha V)\})_\mathfrak{g}\right)$$
$$+ (\bar{\Phi}_i, \mathrm{e}^{2iV}\cdot\Phi^i)_\mathcal{A} + \alpha\left(\int \mathrm{d}^2\theta\ \varepsilon_{ijkl}([\Phi^i,\Phi^j,\Phi^k],\Phi^l)_\mathcal{A} + c.c.\right)$$

Admissible 3-Algebraic Structures

Imposing gauge invariance in the $\mathcal{N}=2$ BLG-like model leads to more freedom.

Demanding gauge invariance in above theory yields the condition:

$$([A,B,C],D)_\mathcal{A} = -([B,A,C],D)_\mathcal{A} = -([A,B,D],C)_\mathcal{A} = ([C,D,A],B)_\mathcal{A}$$

Cherkis, CS, 0807.0808

Generalized metric 3-Lie algebras

$\mathcal{A}$ a real vector space with map $[\cdot,\cdot,\cdot]:\mathcal{A}\times\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ satisfying

$$[A,B,[C,D,E]] = [[A,B,C],D,E]+[C,[A,B,D],E]+[C,D,[A,B,E]] \quad \text{(FI)}$$

and

$$([A,B,C],D)_\mathcal{A} = -([B,A,C],D)_\mathcal{A} = ([C,D,A],B)_\mathcal{A} \quad \text{(Sym)}$$

and a bilinear symmetric map $(\cdot,\cdot)_\mathcal{A}:\mathcal{A}\otimes\mathcal{A}\to\mathbb{R}$ satisfying

$$([A,B,C],D)_\mathcal{A}+(C,[A,B,D])_\mathcal{A} = 0 \quad \text{(Cmp)}$$

Because of the fundamental identity FI, we still have an associated Lie algebra $\mathfrak{g}_{\mathcal{A}} := \mathrm{im}(D_{\mathcal{A}\wedge\mathcal{A}})$ acting on $\mathcal{A}$ by linearly extending

$$D_{A\wedge B}(C) := [A, B, C] \ , \quad A, B, C \in \mathcal{A}$$

Examples for generalized metric 3-Lie algebras:
Clifford algebra $Cl(\mathbb{R}^{2d}, \delta_{ab})$ generated by γ_a, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$.
Define: $\mathcal{C}_{2d}$ as the vector space spanned by the $\gamma_I = \gamma_{[a_1 \dots a_i]}$, endowed with:

$$[\gamma_I, \gamma_J, \gamma_K] := [[\gamma_I, \gamma_J]\gamma_{ch}, \gamma_K] \ , \quad (\gamma_I, \gamma_J)_{\mathcal{A}} = \mathrm{tr}\,(\gamma_I^\dagger \gamma_J)$$

Note that $\mathcal{C}_4 = A_4 \oplus A_4$, as in this case:

$$[\gamma_a, \gamma_b, \gamma_c] = [[\gamma_a, \gamma_b]\gamma_5, \gamma_c] \sim \sum_d \varepsilon_{abcd}\gamma_d$$

Hermitian 3-Lie Algebras

Another generalization of 3-Lie algebras are the Hermitian ones yielding $\mathcal{N}=6$ SUSY.

Alternatively to our way of extending 3-Lie algebras:

Reduce supersymmetry to $\mathcal{N}=6$, i.e. assume the following:

$$\begin{aligned}
\delta\phi^i &= \sqrt{2}\bar{\varepsilon}^{ij}\bar{\psi}_j \ , \\
\delta\bar{\psi}_i &= -\mathrm{i}\sqrt{2}\sigma^\mu\varepsilon_{ij}\nabla_\mu\phi^j + [\phi^j,\phi^k;\bar{\phi}_j]\varepsilon_{ik} + [\phi^j,\phi^k;\bar{\phi}_i]\varepsilon_{jk} \ , \\
\delta A_\mu &= -\mathrm{i}\varepsilon_{ij}\sigma_\mu\phi^i\wedge\psi^j + \mathrm{i}\bar{\varepsilon}^{ij}\sigma_\mu\bar{\phi}_i\wedge\bar{\psi}_j \ .
\end{aligned}$$

where ε^{ij} is in the $\mathbf{6}$ of $SU(4)$. Closure of this algebra implies:

$$[A,B;C] = -[B,A;C] \qquad ([A,B;C],D) = (B,[C,D;A]) \ .$$

$$[[C,D;E],A;B] - [[C,A;B],D;E] - [C,[D,A;B];E] + [C,D;[E,B;A]] = 0 \ .$$

An associated Lie algebra $\mathfrak{g}_\mathcal{A} := \mathrm{im}(D_{\mathcal{A}\wedge\mathcal{A}})$ is induced by

$$D_{A\wedge B}(C) := [C,A;B] \ , \quad A,B,C\in\mathcal{A}$$

This leads to the ABJM model, a Chern-Simons-matter theory.

Aharony, Bergman, Jafferis, Maldacena, 0806.1218

Bagger, Lambert, 0807.0163

Equivalence to Gauge Theories

The above generalizations of 3-Lie algebras can be recast into Lie algebra language.

Recall the BLG Lagrangian:

$$\begin{aligned}\mathcal{L}_{\rm BLG} \;=\; &+\tfrac{1}{2}\varepsilon^{\mu\nu\kappa}\left((A_\mu,\partial_\nu A_\kappa)_{\mathfrak{g}}+\tfrac{1}{3}(A_\mu,[A_\nu,A_\kappa])_{\mathfrak{g}}\right)\\ &-\tfrac{1}{2}(\nabla_\mu X,\nabla^\mu X)_{\mathcal{A}\otimes Cl}+\tfrac{\mathrm{i}}{2}(\bar{\Psi},\Gamma^\mu\nabla_\mu\Psi)_{\mathcal{A}}\\ &+\tfrac{\mathrm{i}}{4}(\bar{\Psi},[X,X,\Psi])_{\mathcal{A}}-\tfrac{1}{12}([X,X,X],[X,X,X])_{\mathcal{A}\otimes Cl}\end{aligned}$$

Up to potential terms, this is an ordinary gauge theory.
What is the relationship between Lie algebras and 3-Lie algebras?
(Medeiros, Figueroa-O'Farrill, Mendez-Escobar, Ritter, 0809.1086)

Unifying picture:

Generalized 3-Lie algebras $\leftrightarrow$ $(\mathfrak{g},V)$ $\mathfrak{g}$: real Lie algebra
V: faithful orthogonal $\mathfrak{g}$-mod.

similar statement for Hermitian 3-Lie algebras.

Current Situation:
It is not clear, if 3-Lie algebras are necessary at all.

Observations:

- 3-Lie algebras too restrictive, only one example: A_4.
- Generalizations lead to less than $\mathcal{N} = 8$ supersymmetry.
- All models can be rewritten as gauge theories.

$\Rightarrow$ We need more input from physics.

Particularly important here: AdS/CFT correspondence

We need some kind of $N \to \infty$ limit, so let's look at representations of (generalized) 3-Lie algebras in terms of matrix algebras.

Classifications of $*$-Algebra Representations of 3-Algebras

Representations on matrix algebras, which are useful for $N \to \infty$, can be constructed.

Representation of metric 3-algebras on $*$-algebras:

Take a $*$- or matrix algebra equipped with a trace form. Construct a 3-bracket on this algebra from matrix products and the involution and use the Hilbert-Schmidt scalar product $(A, B) = \mathrm{tr}\,(A^\dagger B)$.

Classification of all such representations in the real and hermitian case using MuPad done in Cherkis, Dotsenko, CS, 0812.3127

The Real case. $[A,B,C] :=$

$$
\begin{aligned}
\mathrm{I} &: \alpha([[A^*,B],C]+[[A,B^*],C]+[[A,B],C^*]-[[A^*,B^*],C^*]) \\
\mathrm{II} &: \alpha([[A,B^*],C]+[[A^*,B],C]) \\
\mathrm{III} &: \alpha(AB^*-BA^*)C+\beta C(A^*B-B^*A) \\
\mathrm{IV} &: \alpha([[A,B],C]+[[A^*,B^*],C]+[[A^*,B],C^*]+[[A,B^*],C^*]) \\
&\quad +\beta([[A,B],C^*]+[[A^*,B],C]+[[A,B^*],C]+[[A^*,B^*],C^*])\ .
\end{aligned}
$$

The class of examples $\mathcal{C}_{2d}$,

$$[\gamma_a,\gamma_b,\gamma_c] := [[\gamma_a,\gamma_b]\gamma_{ch},\gamma_c]\ ,$$

is contained in III, with $\alpha=\beta=-1$ and the $*$-algebra is the algebra of $d\times d$ matrices.

The Hermitian case. $[A, B, C] :=$

$$\mathrm{I}_\alpha : A, B, C \mapsto \alpha(AC^\dagger B - BC^\dagger A) \; .$$

This is precisely the Hermitian 3-Lie algebra used in Bagger, Lambert, 0807.0163 to obtain the ABJM model in 3-algebra form.

The manifestly supersymmetric actions from above can be used with any such 3-algebra.

Recall the $\mathcal{N}=2$ superfield formulation of the BLG model:

$$\mathcal{L} = \int \mathrm{d}^4\theta\ \kappa\left(\mathrm{i}(V,(\bar{D}_\alpha D^\alpha V))_{\mathfrak{g}} + \tfrac{2}{3}(V,\{(\bar{D}^\alpha V),(D_\alpha V)\})_{\mathfrak{g}}\right)$$
$$+ (\bar{\Phi}_i, \mathrm{e}^{2iV}\cdot\Phi^i)_{\mathcal{A}} + \alpha\left(\int \mathrm{d}^2\theta\ \varepsilon_{ijkl}([\Phi^i,\Phi^j,\Phi^k],\Phi^l)_{\mathcal{A}} + c.c.\right)$$

as well as the $\mathcal{N}=4$ superfield formulation:

$$\int \mu\ \kappa\Big(\mathrm{i}(\mathcal{V},(\bar{\mathcal{D}}_\alpha \mathcal{D}^\alpha \mathcal{V}))_{\mathfrak{g}} + \tfrac{2}{3}(\mathcal{V},\{(\bar{\mathcal{D}}^\alpha \mathcal{V}),(\mathcal{D}_\alpha \mathcal{V})\})_{\mathfrak{g}}\Big) + \left(\bar{\eta}_k, \mathrm{e}^{2\mathrm{i}\mathcal{V}}\cdot\eta_k\right)_{\mathcal{A}}$$

In both cases, $\mathcal{A}$ can also be a generalized or a Hermitian 3-Lie alg.

Outline

- Review part
  - The Nahm equation or D1-D3 branes
  - The Basu-Harvey equation or M2-M5 branes
  - Stacks of flat M2-branes: The BLG model
- Superspace formulations of BLG-like models
  - Manifestly $\mathcal{N} = 2$ supersymmetric formulation
  - Manifestly $\mathcal{N} = 4$ supersymmetric formulation
- Generalized 3-Lie algebras and BLG-like models
  - The structure of generalized 3-Lie algebras
  - The unifying picture by Figueroa-O'Farrill et al.
  - Representations on $*$-algebras
- ▶ The framework of strong homotopy Lie algebras
  - L_∞ algebras and homotopy Maurer-Cartan (hMC) equations
  - The Nahm and the Basu-Harvey equations as hMC equations
  - The SYM equation as hMC equations
  - The BLG equation as hMC equations

L_∞- or strong homotopy Lie algebras

- Introduced by Stasheff (1992) "only way to extend Lie algebras"
- appear in string FT, top. conf. FT, Morse theory

Definition:

R-module $\mathcal{L}$, with family of R-multilinear maps $\mu_n : \mathcal{L}^{\times n} \to \mathcal{L}$ s.t.:

$$\mu_n(x_{\sigma(1)} \dots x_{\sigma(n)}) = \epsilon(\sigma)\mu_n(x_1 \dots x_n)$$

Homotopy Jacobi-type identity:

$$\sum_{i=1}^{n} \sum_{\sigma \in Sh(i,n-i)} (-1)^{i(n+1)}\epsilon(\sigma)\mu_{n-i+1}(\mu_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}, ..., x_{\sigma(n)}) = 0$$

There is also a graded version, then μ_n is of degree $2-n$.

Note: μ_1 is a differential, $\mu_1, \mu_2 \neq 0 \to$ diff. (grad.) Lie algebra

Interestingly, n-Lie algebras are (ungraded) L_∞-algebras

(Hanlon, Wachs 1995, Dzhumadil'daev, math/0202043)

L_∞-algebras and Homotopy Maurer-Cartan Equations

The eom of the BLG model can be reformulated as homotopy Maurer-Cartan equations.

Equations employing L_∞ algebras:

homotopy Maurer-Cartan equation

Given a (graded) L_∞-algebra $\mathcal{L} = \oplus_i \mathcal{L}_i$,

$$\sum_{\ell \geq 0} \frac{(-1)^{\ell(\ell+1)/2}}{\ell!} \mu_\ell(\varphi^{\otimes \ell}) = 0 \ , \quad \varphi \in \mathcal{L}$$

is invariant under the gauge transformations

$$\delta\varphi = -\sum_{\ell \geq 1} \frac{(-1)^{\ell(\ell-1)/2}}{(\ell-1)!} \mu_\ell(\alpha \otimes \varphi^{\ell-1}) \ , \quad \alpha \in \mathcal{L}_0$$

Andrei Losev: "All classical equations of motion are of hMC form."

If only $\mu_1, \mu_2 \neq 0$, then hMC are ordinary Maurer-Cartan eqs.

The following examples are all developed in
C. I. Lazaroiu, D. McNamee, CS and A. Zejak, 0901.3905

The Nahm Equation as hMC Equations

Both the Nahm and the Basu-Harvey equations can trivially be put into hMC form.

Example (1): The Nahm equation

$$\nabla_s X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

with gauge algebra $\mathfrak{su}(N)$. Using $X = \sigma_i X^i$, rewrite as

$$\mathrm{d}_A X + [X, X]\mathrm{d}s = 0$$

Vector space for the L_∞-algebra: $\mathcal{L} := \Omega^\bullet(\mathbb{R}, Cl(\mathbb{R}^3)) \otimes \mathfrak{su}(N)$
Grading arises from $\widetilde{\mathrm{d}s} = 1$, $\widetilde{\sigma_i} = 1$
Higher products reproducing the Nahm equation:

$$\mu_1(X) := \mathrm{d}X \ , \quad \mu_2(A, X) := [A, X] \ , \quad \mu_2(X, X) := [X, X]\mathrm{d}s$$

Higher products taking care of gauge transformations:

$$\mu_1(\lambda) := \mathrm{d}\lambda \ , \quad \mu_2(\lambda, A) := [\lambda, A] \ , \quad \mu_2(\lambda, X) := [\lambda, X]$$

This reproduces both the eom and gauge symmetry correctly.

The Nahm Equation as hMC, Higher Jacobi Identities

Most of the higher Jacobi identities are automatically satisfied.

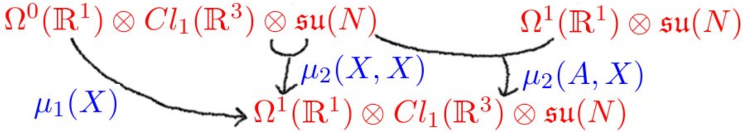
Homtopy Jacobi identity:

$$\sum_{i=1}^{n} \sum_{\sigma \in Sh(i,n-i)} (-1)^{i(n+1)} \epsilon(\sigma) \mu_{n-i+1}(\mu_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0$$

Higher products responsible for equations of motion:

$$\mu_1(X) := \mathrm{d}X \ , \quad \mu_2(A, X) := [A, X] \ , \quad \mu_2(X, X) := [X, X]\mathrm{d}s$$

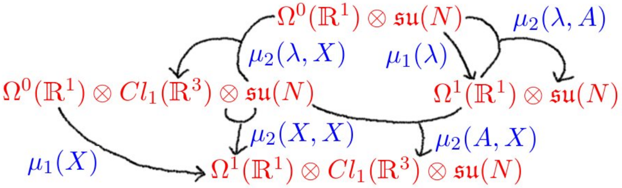
These satisfy the higher Jacobi identities trivially:



All higher products:

$$\mu_1(X) := \mathrm{d}X\ ,\quad \mu_2(A,X) := [A,X]\ ,\quad \mu_2(X,X) := [X,X]\mathrm{d}s$$

$$\mu_1(\lambda) := \mathrm{d}\lambda\ ,\quad \mu_2(\lambda,A) := [\lambda,A]\ ,\quad \mu_2(\lambda,X) := [\lambda,X]$$



The hom. Jacobi identities define the following higher products:

$$\mu_1(\mu_2(\lambda,X))\quad\&\quad\mu_2(\mu_1(\lambda),X)\quad\Rightarrow\quad\mu_2(\lambda,\mu_1(X))$$

$$\mu_2(\mu_2(\lambda,A),X)\quad\&\quad\mu_2(\mu_1(\lambda,X),A)\quad\Rightarrow\quad\mu_2(\lambda,\mu_2(A,X))$$

$$\mu_2(\mu_2(\lambda,X),X)\quad\Rightarrow\quad\mu_2(\lambda,\mu_2(X,X))$$

The Basu-Harvey Equation as hMC Equations

Both the Nahm and the Basu-Harvey equations can trivially be put into hMC form.

Example (2): The Basu-Harvey equation

$$\nabla_s X^i + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0$$

with 3-Lie algebra $\mathcal{A}$ and associated gauge algebra $\mathfrak{g}_{\mathcal{A}}$. Rewrite:

$$\mathrm{d}_A X + \gamma_5[X, X, X]\mathrm{d}s = 0 \quad , \qquad X := \gamma_i X^i$$

Vector space for the L_∞-algebra: $\mathcal{L} := \Omega^\bullet(\mathbb{R}, Cl(\mathbb{R}^4)) \otimes (\mathcal{A} \oplus \mathfrak{g}_{\mathcal{A}})$
Grading arises from $\widetilde{\mathrm{d}s} = 1$, $\widetilde{\gamma_i} = 1$ Higher products:

$$\mu_1(X) := \mathrm{d}X \quad \mu_2(A, X) := [A, X] \quad \mu_3(X, X) := \gamma_5[X, X, X]\mathrm{d}s$$

Higher products taking care of gauge transformations:

$$\mu_1(\lambda) := \mathrm{d}\lambda \quad \mu_2(\lambda, A) := [\lambda, A] \quad \mu_2(\lambda, X) := [\lambda, X]$$

Higher Jacobi identities require us to define further products. This reproduces eom and gauge symmetry correctly.

The Super Yang-Mills Equations as hMC Equations

Because of their second-order nature, the hMC form of the SYM equations is more subtle.

Example (3): The (bosonic part of the) super Yang-Mills eqns:

$$\nabla_\mu F^{\mu\nu} = [X^i, \nabla^\nu X^i] \quad \nabla_\mu \nabla^\mu X^i = [[X^i, X^j], X^j]$$

gauge algebra $\mathcal{A} = \mathfrak{su}(N)$.

New: differential operators of second order. Rewrite $X := X^i \gamma_i$:

$$* \mathrm{d}_A * \mathrm{d}_A A \gamma_{\mathrm{ch}} = \mathrm{tr}_{Cl}([X, \mathrm{d}_A X])\gamma_{\mathrm{ch}} \quad (\Delta_A X)\omega = \gamma_{\mathrm{ch}}[X, \gamma_{\mathrm{ch}}[X, X]]\omega$$

Vector space for the L_∞-algebra: $\mathcal{L} := \Omega^\bullet(\mathbb{R}^{1,p}) \otimes Cl(\mathbb{R}^{9-p}) \otimes \mathcal{A}$

Grading arises from $\widetilde{\mathrm{d}x^\mu} = 1$, $\tilde{\gamma}_i = 1$ Higher products, 1st eq.:

$$\mu_1(A) := -(*\mathrm{d} * \mathrm{d}A)\,\gamma \qquad \mu_2(A, A) := (*[A, *\mathrm{d}A] + *\mathrm{d} * [A, A])\,\gamma\,,$$

$$\mu_3(A, A, A) := (*[A, *[A, A]])\,\gamma \qquad \mu_2(X, X) := \mathrm{tr}_C([X, \mathrm{d}X])\,\gamma$$

$$\mu_3(X, A, X) := \mathrm{tr}_C([X, [A, X]])\,\gamma\,.$$

Higher products, 2nd eq.:

$$\mu_1(X) = -\Delta X\,\omega \qquad \mu_3(X, X, X) = -6\,(\gamma[X, \gamma[X, X]])\,\omega$$

$$\mu_2(A, X) = -([A_\mu, \partial^\mu X]\omega + \partial_\mu[A^\mu, X])\,\omega \qquad \mu_3(A, A, X) = [A_\mu, [A^\mu, X]]\,\omega\,,$$

Gauge sym., higher Jacobi identities, SUSY $\Rightarrow$ higher products.

The BLG Equations as hMC Equations

The eom of the BLG model can be reformulated as homotopy Maurer-Cartan equations.

Example (4): The (bosonic part of the) BLG eqns:

$$\nabla_\mu \nabla^\mu X + \tfrac{1}{2}\Gamma[X, X, \Gamma[X, X, X]] \;=\; 0$$
$$[\nabla_\mu, \nabla_\nu] + \varepsilon_{\mu\nu\kappa}(\,\mathrm{tr}_{\,Cl}(X \wedge (\nabla^\kappa X))) \;=\; 0$$

Start with 3-Lie algebra $\mathcal{A}$ and introduce the module

$$\mathcal{L} := \Omega^\bullet(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_8 \otimes (\mathcal{A} \oplus \mathfrak{g}_{\mathcal{A}})$$

define gradings:

$$\deg(\Omega^0(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_{8,0} \otimes \mathfrak{g}_{\mathcal{A}}) = 0$$
$$\deg(\Omega^1(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_{8,0} \otimes \mathfrak{g}_{\mathcal{A}}) = \deg(\Omega^0(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_{8,1} \otimes \mathcal{A}) = 1$$
$$\deg(\Omega^2(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_{8,0} \otimes \mathfrak{g}_{\mathcal{A}}) = \deg(\Omega^3(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_{8,1} \otimes \mathcal{A}) = 2$$

The fields will live in the following subspaces:

$$A \in \Omega^1(\mathbb{R}^3) \otimes Cl_{8,0} \otimes \mathfrak{g}_{\mathcal{A}} \qquad X \in \Omega^0(\mathbb{R}^3) \otimes Cl_{8,1} \otimes \mathcal{A}$$
$$\lambda \in \Omega^0(\mathbb{R}^3) \otimes Cl_{8,0} \otimes \mathfrak{g}_{\mathcal{A}}$$

The BLG Equations as hMC Equations

The eom of the BLG model can be reformulated as homotopy Maurer-Cartan equations.

BLG equations of motion (bosonic part):

$$\begin{aligned} \nabla_\mu \nabla^\mu X + \tfrac{1}{2}\Gamma[X, X, \Gamma[X, X, X]] &= 0 \\ [\nabla_\mu, \nabla_\nu] + \varepsilon_{\mu\nu\kappa}(\operatorname{tr}_{Cl}(X \wedge (\nabla^\kappa X))) &= 0 \end{aligned}$$

Define the following brackets:

$$\mu_1(A) := \mathrm{d}A \qquad \mu_2(A, A) := [[A \wedge A]] \ ,$$

$$\mu_2(X, X) := *\tau(X \wedge \mathrm{d}X) \qquad \mu_3(A, X, X) := *\tau(X \wedge [A, X])$$

$$\mu_1(X) := \Delta X \omega \qquad \mu_2(A, X) := \partial_\mu [A^\mu, X]\omega + [A_\mu, \partial^\mu X]\omega$$

$$\mu_3(A, A, X) := [A_\mu, [A^\mu, X]]\omega \qquad \mu_5(X^{\otimes 5}) := \Gamma[X, X, \Gamma[X, X, X]]$$

further brackets consistently from gauge symmetry, supersymmetry and homotopy Jacobi identities.

The hMC equations $\sum_{\ell \geq 0} \frac{(-1)^{\ell(\ell+1)/2}}{\ell!}\mu_\ell(\varphi^{\otimes \ell}) = 0$ reproduce the BLG model together with its gauge invariance. (SUSY extension)

Past work:

- Identification of extended 3-algebraic structures
- Classification of categorical matrix representations
- Manifestly $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric formulations of BLG-like models
- Identification of L_∞-algebra structure
- BLG eoms rewritten as homotopy Maurer-Cartan equations

Future directions:

- Are L_∞-algebras useful here? Extendable? Classifications?
- Which 3-algebras yield Hamiltonians of integrable spin chains?
- Extend SUSY models by Yang-Mills term, analyze
- Lift the Nahm/Fourier-Mukai transform to M-theory
- Ultimately: find analogous models for M5 branes

On the Effective Description of Multiple M2-Branes

Christian Sämann

School of Mathematics, Trinity College Dublin

University of Amsterdam, March, 24th 2009