The Geometry of M2-Branes Ending on M5-Branes

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Based on:

- Joshua DeBellis, CS and Richard J. Szabo, arXiv:1001.3275

The Nahm Equation or D1-D3-Branes

In type IIB string theory, monopoles can be seen as D1-branes ending on D3-branes.

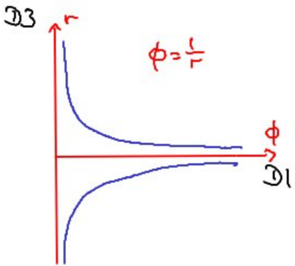
Consider a D3-brane in directions 0123.

A BPS solution to the SYM equations is a magnetic monopole with Higgs field $\phi \sim \frac{1}{r}$: A D1-brane appears.

As they are BPS, one trivially forms a stack of N D1-branes.

From the perspective of the D1-brane, the effective dynamics is described by the Nahm equations:

$$\frac{\mathrm{d}}{\mathrm{d}\phi} X^i + \varepsilon^{ijk}[X^j, X^k] = 0 \ .$$



dim	0	1	2	3	4
$D1$	×				×
$D3$	×	×	×	×	

These equations have the following solution ("fuzzy funnel")

$$X^i = r(\phi) G^i \ , \quad r(\phi) = \frac{1}{\phi} \ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

The Basu-Harvey Equation or M2-M5-Branes

M2 branes ending on M5 branes should be described by Nahm-type equations.

M5-brane in directions 012345:

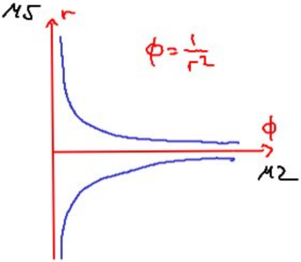
$$G^{mn}\nabla_m\nabla_n X^{a'} = 0$$
$$G^{mn}\nabla_m H_{npq} = 0$$

Ansatz for a soliton:

$$X^{6'} = \phi$$
$$H_{01m} = v_m \quad H_{mnp} = \varepsilon_{mnpq}v^q$$

Solution:

$$H_{01m} \sim \partial_m\phi \quad \phi \sim \frac{1}{r^2}$$



dim	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

Perspective of M2: postulate four scalar fields X^i, satisfying

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^i + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0$$

Basu, Harvey, hep-th/0412310

The Basu-Harvey Equation or M2-M5-Branes

M2 branes ending on M5 branes should be described by Nahm-type equations.

Basu-Harvey equation:

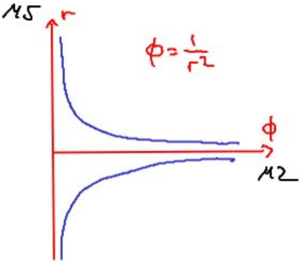
$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^i + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0$$

Solution (similar to D1-D3 case):

$$X^i = r(\phi)G^i \quad r(\phi) = \frac{1}{\sqrt{\phi}}$$

$$G^i = \varepsilon^{ijkl}[G^j, G^k, G^l]$$

Interprete this again as a fuzzy funnel, this time with a fuzzy S^3?



dim	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

Motivating Question

Can one assign geometric meaning to such 3-brackets?

Outline

- Introductory part
  - 3-Lie algebras
  - Nabu-Poisson structures
  - Geometries we will focus on
- Classical quantization
  - Axioms of quantization
  - Berezin-Toeplitz quantization of $\mathbb{C}P^1$
- Generalizations of this quantization procedure
  - Axioms of generalized quantization
  - Quantization of S^4
  - Quantization of $\mathbb{R}^3$
- Conclusions

What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

Basu-Harvey equation:

$$\frac{\mathrm{d}}{\mathrm{d}\phi} X^i + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0 \ , \quad X^i(\phi) \in \mathcal{A}$$

- ▷ $\mathcal{A}$ forms a vector space.
- ▷ $[\cdot,\cdot,\cdot]$ is a totally antisymmetric, linear map $\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \to \mathcal{A}$.

What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

Basu-Harvey equation:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^i + [A_\phi, X^i] + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0\ , \quad X^i \in \mathcal{A}$$

▷ Gauge transformations from inner derivations:

The triple bracket forms a map $\delta : \mathcal{A} \wedge \mathcal{A} \to \mathrm{Der}(\mathcal{A}) =: \mathfrak{g}_\mathcal{A}$ via

$$\delta_{A\wedge B}(C) := [A, B, C]$$

Demand a "3-Jacobi identity," the fundamental identity:

$$\begin{aligned}\delta_{A\wedge B}(\delta_{C\wedge D}(E)) &:= [A, B, [C, D, E]] \\ &= [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]]\end{aligned}$$

The inner derivations form indeed a Lie algebra:

$$[\delta_{A\wedge B}, \delta_{C\wedge D}](E) := \delta_{A\wedge B}(\delta_{C\wedge D}(E)) - \delta_{C\wedge D}(\delta_{A\wedge B}(E))$$

Bracket closes due to fundamental identity.

Nambu-Poisson structures are special n-Lie algebra structures on $\mathcal{C}^\infty(\mathcal{M})$.

Definition

An n-Lie algebra is a vector space endowed with a totally antisymmetric, n-ary map satisfying the fundamental identity, an "n-Jacobi identity".

Definition

A Nambu-Poisson structure on a smooth manifold $\mathcal{M}$ is a totally antisymmetric, n-ary map $\mathcal{C}^\infty(\mathcal{M})^{\wedge n} \to \mathcal{C}^\infty(\mathcal{M})$ satisfying the fundamental identity

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \{\{f_1, \dots, f_{n-1}, g_1\}, \dots, g_n\} + \cdots + \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_n\}\}$$

as well as the generalized Leibniz rule

$$\{f_1\, f_2, f_3, \dots, f_{n+1}\} = f_1\, \{f_2, \dots, f_{n+1}\} + \{f_1, \dots, f_{n+1}\}\, f_2 \ .$$

A_4

Consider the vector space $\mathbb{R}^4$ with basis $\tau_1, \ldots, \tau_4$.
Then define the bracket $[\cdot,\cdot,\cdot]$ as the linear extension of

$$[\tau_a, \tau_b, \tau_c] = \sum_d \varepsilon_{abcd}\tau_d \quad .$$

Nambu-Poisson structure on S^3

Consider S^3 embedded into $\mathbb{R}^4$ with cartesian coordinates $x^1, \ldots, x^4$. Define the bracket $\{\cdot,\cdot,\cdot\}$ as the extension via linearity and generalized Leibniz rule of

$$\{x^\mu, x^\nu, x^\kappa\} = \sum_\lambda \varepsilon_{\mu\nu\kappa\lambda}x^\lambda \quad .$$

Geometries we will focus on

Aim: To make sense of fuzzy S^3 and the geometry of the Nambu-Heisenberg algebra.

We will focus on the two most obvious geometries:

- D1-D3 yields fuzzy S^2.

  ▷ What is the definition of fuzzy S^3 appearing for M2-M5?

  This has many interesting implications: BLG, ...

- Best-known NC geometry: Moyal plane $[x^1, x^2] \sim \mathbb{1}$.

  ▷ What is the NC geometry of $[x^1, x^2, x^3] \sim \mathbb{1}$?

  This Nambu-Heisenberg algebra was found as the WV equation of M5-branes in certain backgrounds

  C. Chu and D. J. Smith, arXiv:0901.1847

Axioms of Quantization

Quantization is nontrivial and far from being fully understood.

Classical level: states are points on a Poisson manifold $\mathcal{M}$.
observables are functions on $\mathcal{M}$.

Quantum level: states are rays in a complex Hilbert space $\mathscr{H}$.
observables are hermitian operators on $\mathscr{H}$.

Full Quantization

A full quantization is a map $\hat{\ } : \mathcal{C}^\infty(\mathcal{M}) \to \mathrm{End}\,(\mathscr{H})$ satisfying

1. $f \mapsto \hat{f}$ is linear over $\mathbb{C}$, $f = f^* \Rightarrow \hat{f} = \hat{f}^\dagger$.
2. the constant function $f = 1$ is mapped to the identity on $\mathscr{H}$.
3. Correspondence principle: $\{f_1, f_2\} = g \Rightarrow [\hat{f}_1, \hat{f}_2] = \hat{g}$.
4. The quantized coordinate functions act irreducibly on $\mathscr{H}$.

Problem:

Groenewold-van Howe: no full quant. for $T^*\mathbb{R}^n$ or S^2 (T^2 OK)

Loopholes to the obstructions to full quantizations

There are three weaker possible weakenings to the set of axioms for quantization.

Three approaches to weaken the axioms of a full quantization:

- Drop irreducibility
- Quantize a subset of $\mathcal{C}^\infty(\mathcal{M})$
- Correspondence principle applies only to $\mathcal{O}(\hbar)$

The first two yield prequantization and geometric quantization.
The last approach leads eventually to deformation quantization.

We will use Berezin quantization (or fuzzy geometry),
a hybrid of geometric and deformation quantization.

Berezin Quantization of $\mathbb{C}P^1 \simeq S^2$

The fuzzy sphere is the Berezin quantization of $\mathbb{C}P^1$.

Hilbert space

$\mathscr{H}$ is the space of global holomorphic sections of a certain line bundle: $\mathscr{H} = H^0(\mathcal{M}, L)$. For $\mathcal{M} = \mathbb{C}P^1$: $L := \mathcal{O}(k)$.

$$\mathscr{H}_k \cong \mathrm{span}(z_{\alpha_1} \ldots z_{\alpha_k}) \cong \mathrm{span}(\hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_k}|0\rangle)$$

Coherent states

For any $z \in \mathcal{M}$: coherent st. $|z\rangle \in \mathscr{H}$. Here: $|z\rangle = \frac{1}{k!}(\bar{z}_\alpha \hat{a}^\dagger_\alpha)^k |0\rangle$.

Quantization

Quantization is the inverse map on the image $\Sigma = \sigma(\mathcal{C}^\infty(\mathcal{M}))$ of

$$f(z) = \sigma(\hat{f}) = \int \frac{\omega^n}{n!} \frac{|z\rangle\langle z|}{\langle z|z\rangle} \hat{f} \ .$$

Axioms of Generalized Quantization

We propose a generalization of the quantization axioms to Nambu-Poisson manifolds.

Problem is notoriously difficult, and many people tried to extend geometric quantization. Berezin quantization should be easier. Keep: a complex Hilbert space $\mathscr{H}$ and $\mathsf{End}\,(\mathscr{H})$ as observables.

Generalized quantization axioms

A full quantization is a map $\hat{-} : \Sigma \to \mathsf{End}\,(\mathscr{H}),\ \Sigma \subset \mathcal{C}^\infty(M)$ satisfying

1. $f \mapsto \hat{f}$ is linear over $\mathbb{C}$, $f = f^* \Rightarrow \hat{f} = \hat{f}^\dagger$.
2. the constant function $f = 1$ is mapped to the identity on $\mathscr{H}$.
3. Correspondence principle:

$$\lim_{\hbar \to 0} \left\| \frac{\mathrm{i}}{\hbar} \sigma\big([\hat{f}_1, \ldots, \hat{f}_n]\big) - \{f_1, \ldots, f_n\} \right\|_{L^2} = 0$$

If $\mathcal{M}$ is a Poisson manifold, this holds for Berezin quantization.

A natural n-Lie bracket

Truncating the Nambu-Poisson algebra allows for an unusual n-Lie bracket.

On the algebra of polynomials, one can often truncate the Nambu-Poisson algebra to obtain a corresponding n-Lie algebra.

Then one can introduce

$$[\hat{A}_1, \dots, \hat{A}_n] := \sigma^{-1}(-\mathrm{i}\hbar\{\sigma(\hat{A}_1), \dots, \sigma(\hat{A}_n)\}_K) ,$$

and the correspondence principle holds always automatically.

Interesting is the comparison of this to the totally antisymmetric operator product.

Observation:
Using the Clifford algebra $Cl(\mathbb{R}^4)$, we find embedding $S^4 \hookrightarrow \mathbb{C}P^3$:

$$x^\mu = \frac{R}{|z|^2} \gamma^\mu_{\alpha\beta} \bar{z}^\alpha z^\beta \ , \quad \sum_\mu x^\mu x^\mu = R^2 \ .$$

Embedding not holomorphic, otherwise: factor out ideal:

$$\mathcal{M} = \{z \in \mathbb{C}P^n | f(z) = 0\} \to \mathscr{H}_\mathcal{M} = \{|\mu\rangle \in \mathscr{H}_{\mathbb{C}P^n} | \hat{f}|\mu\rangle = 0\}$$

CS, hep-th/0612173

Make the following idea rigorous:

$$\hat{x}^\mu := \frac{R}{|z|^2} \gamma^\mu_{\alpha\beta} \frac{\hat{a}^\dagger_\alpha \hat{a}^\dagger_{\gamma_1} \dots \hat{a}^\dagger_{\gamma_{k-1}} |0\rangle\langle 0| \hat{a}_\beta \hat{a}_{\gamma_1} \dots \hat{a}_{\gamma_{k-1}}}{k!}$$

This satisfies $\sum_\mu \hat{x}^\mu \hat{x}^\mu \sim R^2 \mathbb{1}$ and on linear level is identical to the totally antisymmetric operator product. This quantization yields the Guralnik/Ramgoolam spheres. Hyperboloids, ...

Z. Guralnik and S. Ramgoolam, hep-th/0101001

Quantization of $\mathbb{R}^3$

The quantized Nambu-Heisenberg algebra corresponds to the space $\mathbb{R}^3_\lambda$.

What is the geometry of $[\hat{x}, \hat{y}, \hat{z}] = -\mathrm{i}\,\hbar\,\mathbb{1}$?

No 3-bracket ensuring the correspondence principle.
$\Rightarrow$ 3-algebra structure only at linear level.

One possible interpretation as $\mathbb{R}^3_\lambda$:
Take a fuzzy sphere with Hilbert space $H^0(\mathbb{C}P^1, \mathcal{O}(k))$. Define:

$$[\hat{x}^1, \hat{x}^2, \hat{x}^3] = \sum_{i,j,k} \varepsilon^{ijk} \hat{x}^i \hat{x}^j \hat{x}^k = -\mathrm{i}\,\frac{6R^3}{k}\,\mathbb{1}_{\mathscr{H}_k}$$

Radius of this fuzzy sphere: $R_{F,k} = \sqrt{1 + \frac{2}{k}}\;\sqrt[3]{\frac{\hbar\,k}{6}}$.
Now "discretely foliate" $\mathbb{R}^3$ by fuzzy spheres. $\Rightarrow \mathbb{R}^3_\lambda$.

A. B. Hammou, M. Lagraa, M. M. Sheikh-Jabbari, hep-th/0110291

Done:

- Naive extension of quantization to Nambu-Poisson manifolds.
- NC interpretation of fuzzy 3-funnel and NH algebra.
- M5-brane geometry in M2-M5 + background: $\mathbb{R}^{1,2}_\lambda \times \mathbb{R}^3_\lambda$.
- All spheres, hyperboloids and superspheres can be quantized.

Future directions:

- Quantization of S^3 via gerbes.
- Understand Nahm tranform for M2-M5.

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