Matrix Models and Scalar Field Theories on Noncommutative Spaces

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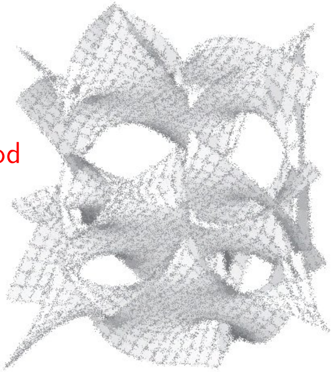
Based on:

- D. O'Connor and CS, JHEP 08 (2007) 066
- CS, SIGMA 6 (2010) 50
- M. Ihl, C. Sachse and CS, JHEP 03 (2011) 091

Outline

1. Partition functions in Matrix Models
2. The Fuzzy Sphere
3. Fuzzy Scalar Field Theory
4. Perturbative Expansion
5. Large N Limit & Saddle Point Method
6. Comparison to Known Results
7. Modification of the Model
8. Conclusions



In QFT, we have to compute path integrals/partition functions:

$$Z = \int \mathscr{D}\varphi \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} S[\varphi]}$$

Various problems with this:

- Wick rotation
- Have to introduce cut-off/regulator
- Perturbative expansions do not see topological effects
- Degeneracies for gauge equivalences

However, there are examples where Z can be computed.

Focus here: $d = 0$.

One-Hermitian Matrix Model

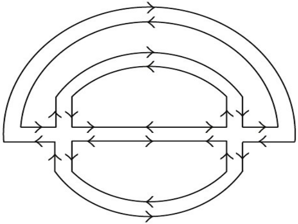
$$Z \; = \; \int \mathrm{d}\mu_D(\Phi)\; \mathrm{e}^{-\,\mathrm{tr}\,\left(r\,\Phi^2 + g\,\Phi^4\right)}$$

Ingredients:

- Φ: field taking values in Hermitian $N \times N$ matrix
- $S[\Phi] := \mathrm{tr}\,\left(r\,\Phi^2 + g\,\Phi^4\right)$: action
- $\mathrm{d}\mu_D(\Phi) := \prod_{i\leq j} \mathrm{d}\Re(\Phi_{ij}) \prod_{i<j} \mathrm{d}\Im(\Phi_{ij})$: Dyson measure
- Large degeneracy: Invariance of $\mathrm{tr}\,(\Phi^n)$ under $\Phi \to \Omega\Phi\Omega^{-1}$

First approach: Perturbation theory

- Φ^2-term gives a "propagator" $\frac{1}{r}$
- Φ^4-term a vertex with coupling g
- Double line Feynman diagrams.



(With Φ^3, used in dynamic triangulation of surfaces.)

The Hermitian Matrix Model

The (One-)Hermitian Matrix Model can be solved in various ways.

Now: Exact solution for

$$Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(r\,\Phi^2 + g\,\Phi^4\right)}$$

Use invariance: $\Phi = \Omega\Lambda\Omega^\dagger$, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$.

$$\int \mathrm{d}\mu_D(\Phi) = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \underbrace{\prod_{j>k} (\lambda_j - \lambda_k)^2}_{\text{Vandermonde determinant}} \int \underbrace{\mathrm{d}\mu_H(\Omega)}_{\text{Haar measure}}$$

Absorbing the constant factor $\int \mathrm{d}\mu_H(\Omega)$, we obtain:

$$Z = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i\ \mathrm{e}^{-2\sum_{i>j} \ln|\lambda_i - \lambda_j| r \sum_i \lambda_i^2 + g \sum_i \lambda_i^4}$$

I) Coulomb-gas of repulsive eigenvalues II) Degeneracy gone.

Continue with saddle point, orthogonal polynomials, etc.

Motivation for Studying Fuzzy Geometry

Fuzzy spaces appear naturally in string theory. They also can serve as regulators of QFTs.

Planck-Scale Structure of Spacetime

Smooth structure of spacetime probably not to arbitrary scales.
The most prominent modifications: SUSY and Noncommutativity.
Fuzzy Geometry: NC on compact symplectic Riemannian spaces arise naturally in string theory

Regularization of Field Theories

Field theories on fuzzy spaces: finite-dimensional matrix models.
QFTs are finite and path integrals well-defined.
Advantages over lattice approach:
Isometries preserved, no fermion doubling, analytical handle
Numerical Simulations are easily done

The Fuzzy Sphere

Idea: Truncate the spherical harmonics at a certain angular momentum.

As usual, do not quantize space itself, but algebra of functions.

Here: Spherical harmonics Y_{lm} with $l = 0, .., \infty$ and $m = -l, ..., l$.

Quantization: Truncate angular momentum $l \leq L$

Result: The space becomes fuzzy

Multiplication will not close any more:

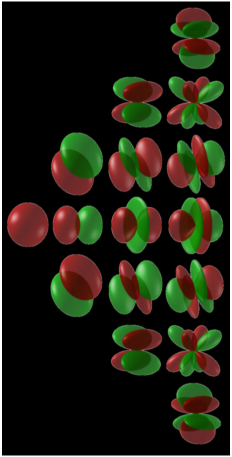
$$Y_{l_1 ...} Y_{l_2 ...} = Y_{l_1 + l_2 ...} + ...$$

However: can deform product to star product

$$[x^i \overset{\star}{,} x^j] \sim \mathrm{i}\varepsilon^{ijk} x^k \ ,$$

where $x^i \in \mathbb{R}^3 \supset S^2$ yields a closed, truncated algebra.

Note: Truncated spherical harmonics can be arranged in matrix.



$$S^2 \cong \mathbb{C}P^1$$

Recall the Hopf fibration

$$0 \to \mathsf{U}(1) \to S^3 \to \mathbb{C}P^1 \to 0 \ .$$

Functions depending on $Y_{lm}, l \leq L$ can be written in terms of homogeneous coordinates z_α ($x^i \sim \bar{z}_\alpha \sigma^i_{\alpha\beta} z_\beta$) on $\mathbb{C}P^1$ as

$$f(x) = f(z) = \sum_{\alpha_i,\beta_i} f^{\alpha_1 \dots \alpha_L \beta_1 \dots \beta_L} \ \frac{z_{\alpha_1} \dots z_{\alpha_L} \bar{z}_{\beta_1} \dots \bar{z}_{\beta_L}}{|z|^{2L}}$$

with $\alpha_i, \beta_i = 1, 2$. (Reduction: $\mathbb{C}^2 \to S^3 \to \mathbb{C}P^1$)

Quantization as in flat case, $(z_\alpha, \bar{z}_\beta) \to (\hat{a}^\dagger_\alpha, \hat{a}_\beta)$:

$$\hat{f}(x) = \hat{f}(z) = \sum_{\alpha_i,\beta_i} f^{\alpha_1 \dots \alpha_L \beta_1 \dots \beta_L} \ \hat{a}^\dagger_{\alpha_1} \dots \hat{a}^\dagger_{\alpha_L} |0\rangle\langle 0| \hat{a}_{\beta_1} \dots \hat{a}_{\beta_L}$$

In this way ("Berezin quantization"), also fuzzy $X \hookrightarrow \mathbb{C}P^n$...

Fuzzy Scalar Field Theory: Definition.

Scalar field theory on the fuzzy sphere is a finite hermitian matrix model

Quantized algebra of functions on the fuzzy sphere:

$$\mathrm{span}(\ \hat{a}^\dagger_{\alpha_1}\ldots\hat{a}^\dagger_{\alpha_L}|0\rangle\langle 0|\hat{a}_{\beta_1}\ldots\hat{a}_{\beta_L}\ ) \cong \mathrm{Mat}(L+1)$$

Laplacian and integration on the fuzzy sphere:

$$\mathcal{L}_i = \mathrm{i}\varepsilon_{ijk}x^j\partial_k \to [L_i,\cdot]\ ,\quad \Delta \to C_2\ ,\quad \int_{S^2}\mathrm{d}A\ f\ \to\ \frac{4\pi R^2}{N}\,\mathrm{tr}\,(\hat{f})$$

The action of real scalar field theory on the fuzzy sphere:

$$S\ \sim\ \mathrm{tr}\ \left(\beta[L_i,\Phi][L_i,\Phi] + r\,\Phi^2 + g\,\Phi^4\right)$$

We define the partition function

$$Z\ =\ \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\,\mathrm{tr}\,\left(\beta[L_i,\Phi][L_i,\Phi]+r\,\Phi^2+g\,\Phi^4\right)}$$

with the Dyson measure

$$\mathrm{d}\mu_D(\Phi)\ =\ \prod_{i\leq j}\mathrm{d}\Re(\Phi_{ij})\prod_{i<j}\mathrm{d}\Im(\Phi_{ij})$$

Fuzzy Scalar Field Theory vs. HMMs

Fuzzy scalar field theory is significantly harder than matrix models usually considered.

Compare Fuzzy Scalar Field Theory

$$Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(\beta[L_i,\Phi][L_i,\Phi]+r\,\Phi^2+g\,\Phi^4\right)}$$

to One-Hermitian Matrix Model:

$$Z \ = \ \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(r\,\Phi^2+g\,\Phi^4\right)}$$

Diagonalization trick $\Phi = \Omega\Lambda\Omega^\dagger$ does not work due to L_i.

State of the art: Hermitian matrix model with one external matrix

$$Z \ = \ \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(V(A\Phi)+r\,\Phi^2+g\,\Phi^4\right)}$$

Solution: splitting $\Phi = \Omega\Lambda\Omega^\dagger$, as well as character expansion

$$\exp(\operatorname{tr}(V(A\Phi))) = \sum_\rho f_\rho \chi_\rho(A\Phi)$$

Itzykson and Di Francesco, Ann. Poincare 59 (1993) 117

Fuzzy Scalar Field Theory: Simplifications

Symmetry arguments yield simplifications of our model.

Fuzzy scalar field theory: $Z = \int \mathrm{d}\mu_D(\Phi)\, \mathrm{e}^{-\operatorname{tr}\left(\beta[L_i,\Phi][L_i,\Phi]+r\,\Phi^2+g\,\Phi^4\right)}$

$$N = 2: \; Z = \int \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \; (\lambda_1 - \lambda_2)^2 \; \mathrm{e}^{-\left(\beta(\lambda_1-\lambda_2)^2 + r(\lambda_1^2+\lambda_2^2) + g(\lambda_1^4+\lambda_2^4)\right)}$$

Symmetries:

1. $\mathrm{d}\mu_D(\Phi) = \mathrm{d}\mu_D(\Omega\Phi\Omega^\dagger) \Rightarrow \int \mathrm{d}\mu_D(\Phi)\; \mathrm{e}^{-S} = \int \mathrm{d}\mu_D(\Phi)\; \mathrm{e}^{-S_0}$

$$S_0 = \sum_n s_n \operatorname{tr}(\Phi^n) + \sum_{n,m} s_{nm} \operatorname{tr}(\Phi^n) \operatorname{tr}(\Phi^m) + \ldots$$

2. $\mathrm{d}\mu_D(\Phi)\; f(\Phi) \sim \mathrm{d}^{N^2}\Phi^\mu \; f(\Phi^\mu \tau_\mu) \;\; \Rightarrow \;\; S_0 = \sum_n s_n \left(\operatorname{tr}(\Phi^2)\right)^n$

3. $[L_i, \mathbb{1}] = 0, \lambda \leftrightarrow -\lambda \;\; \Rightarrow \;\; \operatorname{tr}([L_i,\Phi]^2) \sim \left(\sum_{i>j} (\lambda_i - \lambda_j)^{2m_k}\right)^{n_l}$

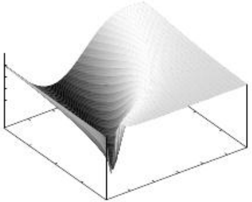
Perturbative expansion: Motivation

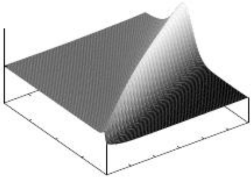
A high temperature expansion yields a useful approximation.

Idea: Treat the kinetic term $\mathrm{e}^{-\operatorname{tr}(\beta[L_i,\Phi][L_i,\Phi])}$ perturbatively.

Motivation:

- Similar to character expansion
- High temperature expansion successfully used on the lattice.
- Specific heat up to $\mathcal{O}(\beta^8)$ for $N = 2$:





- Group theory allows everything else to be treated exactly.

Perturbative expansion: Principles

The angular variables can be integrated out in the perturbative series.

$$Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(\beta[L_i,\Phi][L_i,\Phi]+r\,\Phi^2+g\,\Phi^4\right)}$$

Introduce generators τ_a of $\mathfrak{u}(N)$ such that $\Phi = \Phi^a\tau_a$.

Kinetic term of the action:

$$\operatorname{tr}\,(\Phi[L_i,[L_i,\Phi]]) = \underbrace{\operatorname{tr}\,(\tau_a[L_i,[L_i,\tau_b]])}_{=:K_{ab}}\ \operatorname{tr}\,(\Phi\,\tau^a)\ \operatorname{tr}\,(\Phi\,\tau_b)$$

Expand exponential:

$$\mathrm{e}^{-\beta\operatorname{tr}(\Phi[L_i,[L_i,\Phi]])} = 1 - \beta K_{ab}\Phi_a\Phi_b + \frac{\beta^2}{2}(K_{ab}\Phi_a\Phi_b)^2 - \frac{\beta^3}{6}(K_{ab}\Phi_a\Phi_b)^3 + \mathcal{O}(\beta^4)\ ,$$

Need to compute:

$$\mathcal{I}_k\ :=\ \int \mathrm{d}\mu_H(\Omega)\prod_{i=1}^{k} K_{a_ib_i}\operatorname{tr}\,(\Omega\Lambda\Omega^\dagger\tau_{a_i})\operatorname{tr}\,(\Omega\Lambda\Omega^\dagger\tau_{b_i})$$

Perturbative expansion: Principles

The angular variables can be integrated out in the perturbative series.

Example:

$$\mathcal{I}_2 := \int \mathrm{d}\mu_H(\Omega) K_{a_1 b_1} K_{a_2 b_2} \,\mathrm{tr}\,(\Omega\Lambda\Omega^\dagger \tau_{a_1})\,\mathrm{tr}\,(\Omega\Lambda\Omega^\dagger \tau_{b_1}) \times \\ \times\ \mathrm{tr}\,(\Omega\Lambda\Omega^\dagger \tau_{a_2})\,\mathrm{tr}\,(\Omega\Lambda\Omega^\dagger \tau_{b_2})$$

Use $\mathrm{tr}\,(A)\,\mathrm{tr}\,(B) = \mathrm{tr}\,(A \otimes B)$ and $AB \otimes CD = (A \otimes C)(B \otimes D)$:

$$\mathcal{I}_2 = \int \mathrm{d}\mu_H(\Omega)\, K_{a_1 b_1} K_{a_2 b_2} \times \\ \mathrm{tr}\,\Big((\Omega \otimes \Omega \otimes \Omega \otimes \Omega)(\otimes^4 \Lambda)(\otimes^4 \Omega^\dagger)(\tau_{a_1} \otimes \tau_{b_1} \otimes \tau_{a_2} \otimes \tau_{b_2})\Big)$$

Orthogonality relation:

$$\int \mathrm{d}\mu_H(\Omega) \;\; [\rho(\Omega)]_{ij}\; [\rho^\dagger(\Omega)]_{kl} = \frac{1}{\dim(\rho)}\delta_{il}\delta_{jk}$$

Decompose into irreducible representations of $\mathfrak{su}(N)$!

Perturbative expansion: Results I

Up to $\mathcal{O}(a^3)$, the perturbative expansion is easily doable.

Explicit results:

$$\mathcal{I}_1 = \frac{\mathrm{tr}\,(K)}{N^2-1}\,\mathrm{tr}\,(\Lambda^2) - \frac{\mathrm{tr}\,(K)}{N^3-N}\,\mathrm{tr}\,(\Lambda)^2\,,$$

$$\begin{aligned}
\mathcal{I}_2 =& \frac{10\,\mathrm{tr}\,(K)\left(-2\left(1+N^2\right)+\mathrm{tr}\,(K)\right)+4\left(3-2N^2\right)\mathrm{tr}\left(K^2\right)}{N\left(-36+N^2\left(-7+N^2\right)^2\right)}\,\mathrm{tr}\,(\Lambda^4)+\\
&\frac{40\left(2+2N^2-\mathrm{tr}\,(K)\right)\mathrm{tr}\,(K)+16\left(-3+2N^2\right)\mathrm{tr}\left(K^2\right)}{N^2\left(-36+N^2\left(-7+N^2\right)^2\right)}\,\mathrm{tr}\,(\Lambda^3)\,\mathrm{tr}\,(\Lambda)+\\
&\frac{20\left(-3+2N^2\right)\mathrm{tr}\,(K)+\left(30-14N^2+N^4\right)\mathrm{tr}\,(K)^2+2\left(18-6N^2+N^4\right)\mathrm{tr}\left(K^2\right)}{N^2\left(-36+N^2\left(-7+N^2\right)^2\right)}\,\mathrm{tr}\,(\Lambda^2)^2-\\
&\frac{2\left(100\,\mathrm{tr}\,(K)+\left(-14+N^2\right)\mathrm{tr}\,(K)^2+2\left(6+N^2\right)\mathrm{tr}\left(K^2\right)\right)}{N\left(-36+N^2\left(-7+N^2\right)^2\right)}\,\mathrm{tr}\,(\Lambda^2)\,\mathrm{tr}\,(\Lambda)^2+\\
&\frac{100\,\mathrm{tr}\,(K)+\left(-14+N^2\right)\mathrm{tr}\,(K)^2+2\left(6+N^2\right)\mathrm{tr}\left(K^2\right)}{N^2\left(-36+N^2\left(-7+N^2\right)^2\right)}\,\mathrm{tr}\,(\Lambda)^4
\end{aligned}$$

$$\mathcal{I}_3 = \ldots$$

For $\mathcal{I}_3$, 76 Young tableaux to be considered $\Rightarrow$ Mathematica.

Notice terms $\mathrm{tr}\,(\Lambda^n)^m$.

Perturbative expansion: Results II

Up to $\mathcal{O}(a^3)$, the perturbative expansion is easily doable.

The terms $\mathrm{tr}\,(\Lambda^n)^m$ recombine to a multitrace matrix model:

$$\int \mathrm{d}\mu_H(\Omega)\,\mathrm{e}^{-\,\mathrm{tr}\,(\beta[L_i,\Phi][L_i,\Phi])} = \int \mathrm{d}\mu_H(\Omega)\,\mathrm{e}^{-\sum_{m,n}\alpha_{m,n}(a,N)\,\mathrm{tr}\,(\Phi^m)^n}$$

and the action gets replaced:

$$\mathrm{tr}\,\left(\beta[L_i,\Phi][L_i,\Phi] + r\,\Phi^2 + g\,\Phi^4\right) \rightarrow \mathrm{tr}\,\left(r\,\Phi^2 + g\,\Phi^4\right) + \sum_{m,n}\alpha_{m,n}(a,N)\,\mathrm{tr}\,(\Phi^m)^n$$

Many consistency checks available, e.g. comparison with $N = 2$.

Diagonalization trick $\Phi = \Omega\Lambda\Omega^\dagger$ now works again!

Saddle point approximation

The saddle point approximation gives a rought picture of the actual result.

To solve: take the large N-limit and perform saddle point approx.

Rewrite: $\lambda_i \to \lambda(\frac{i}{N}) = \lambda(x)$, $0 < x < 1$, $\sum_{i=0}^{N} \to N \int_0^1 \mathrm{d}x$

Rescale: $\beta = N^{\theta_\beta}\tilde{\beta}$, $r = N^{\theta_r}\tilde{r}$, $g = N^{\theta_g}\tilde{g}$, $\lambda(x) = N^{\theta_\lambda}\tilde{\lambda}(x)$

Partition function: $Z = \int \mathscr{D}\lambda \ \exp(-N^2\tilde{S})$

With moments $c_n := \int \mathrm{d}x\, \lambda^n(x)$, we have the action:

$$\beta\tilde{S} = \beta N^3(c_2 - c_1^2) - \beta^2 \frac{N^4}{3}\left(c_1^2 - c_2\right)^2 - \beta^3 \frac{4N^5}{27}\left(2c_1^3 - 3c_1 c_2 + c_3\right)^2 + \beta N r c_2 + \beta N g c_4 - N^2 \int \mathrm{d}x\,\mathrm{d}y\, \log|\lambda(x) - \lambda(y)|$$

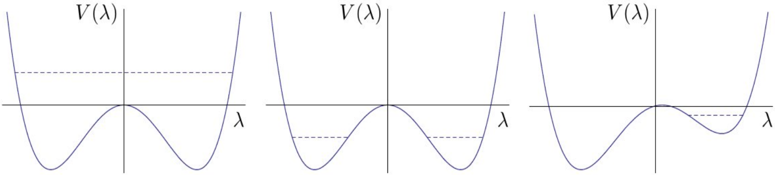
Analysis of the Phase Diagram

Contrary to the hermitian matrix model, there are three phases.

Recall:

- Eigenvalues form one-dimensional Coulomb gas.
- Eigenvalue distribution affects potential.

Phases:



symmetric single cut   symmetric double cut   asymmetric single cut

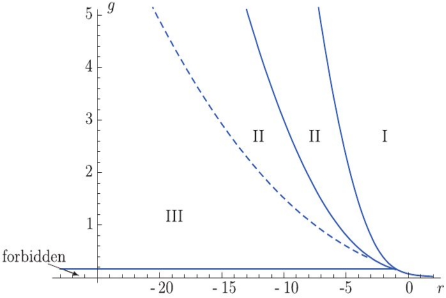
Can compute the classical eigenvalue distribution for each case.

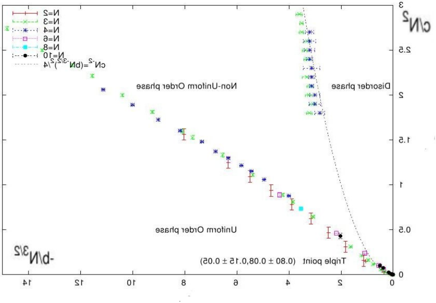
For example: symmetric single cut $[-\delta,\delta]$ up to $\mathcal{O}(\beta^3)$: $u(\tilde{\lambda}) =$

$$\left(4\tilde{r}-\tilde{\beta}+12\pi\tilde{\beta}^2c_2+4\left(\tilde{g}+\tfrac{\pi\tilde{\beta}^2}{2}\right)\delta^2+8\left(\tilde{g}+\tfrac{\pi\tilde{\beta}^2}{2}\right)\tilde{\lambda}^2\right)\sqrt{\delta^2-\tilde{\lambda}^2}$$

Contrary to the hermitian matrix model, there are three phases.

Phases from existence bounds and choosing lowest free energy:





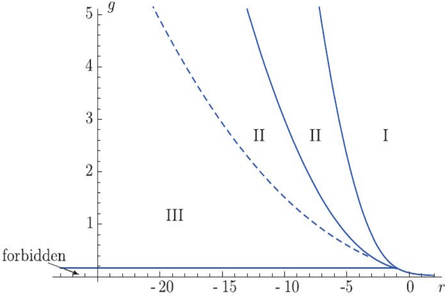
Analytic results

Numerical results

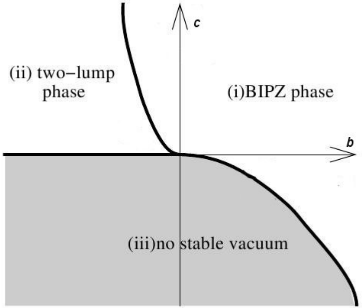
Flores, O'Connor, Martin, hep-th/0601012,
Panero, hep-th/0608202

Comparison with the Matrix Model Phase Diagram

The matrix model phase diagram suggests two phases.



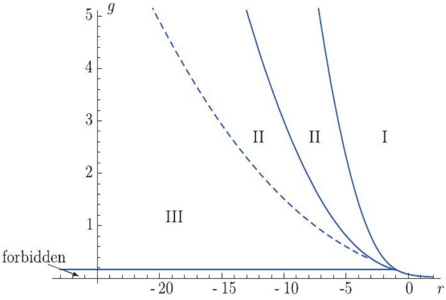
Fuzzy Scalar Field Theory



Hermitian Matrix Model

How well does our model approximate ϕ^4-theory on $\mathbb{R}^2$?

$$Z = \int \mathscr{D}\phi \; \mathrm{e}^{-\int \mathrm{d}^2 x \frac{1}{2}(\nabla\phi)^2 + b\phi^2 + c\phi^4} \;, \qquad \frac{c}{b} = \mathrm{const.}$$



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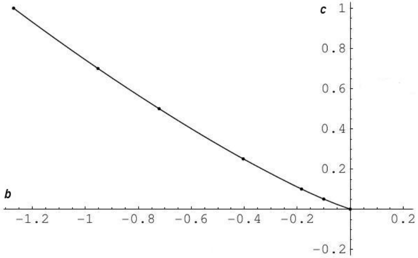

Scalar Field Theory

proof of existence: [Glimm, Jaffe, Spencer, 1974/1975]
exact shape numerically: [Loinaz, Willey, hep-lat/9712008],
confirmed by [Lee, hep-th/9811117]

- In ϕ^4-theory on $\mathbb{R}^2_\theta$:
  New phase predicted [Gubser, Sondhi, hep-th/0006119], analytically confirmed [Chen, Wu, hep-th/0110134] and numerically confirmed [Ambjorn, Catterall, hep-th/0209106].
- Indications for the new phase also found by regularization ϕ^4-theory with fuzzy spaces by [Steinacker, hep-th/0501174].
- Removal of new phase would be an indicator of successful regularization of commutative ϕ^4-theory.

$\Rightarrow$ We are regularizing ϕ^4 on the Moyal plane $\mathbb{R}^2_\theta$.

Modification of the model

The proposed modification of the model moves the triple point in the right direction.

Modify the action, assuming momentum-dependent wave function regularization $\mathcal{Z}_L(C_2) \approx 1 + \kappa C_2$:

$$\tilde{S} = \operatorname{tr}\left(\beta \Phi (C_2 + \kappa C_2 C_2)\Phi + r\,\Phi^2 + g\,\Phi^4\right)$$

Dolan, O'Connor, Presnajder

This implies the following modification in our analysis

$$K_{ab} \to \check{K}_{ab} := K_{ab} + \kappa K_{ac} K_{cb} \quad \text{and} \quad \check{\beta} = \beta(1 + \tfrac{2}{3}\kappa(N^2 - 1))$$

Rescaling of κ to keep highest order term yields $\tilde{\check{\beta}} = \beta(1 + \frac{2}{3}\tilde{\kappa})$.

$\Rightarrow$ The triple point moves off to infinity for increasing $\tilde{\kappa}$.

We discussed the following:

- Matrix models and how to solve them
- The Fuzzy/Berezin quantized sphere
- Scalar field theories on fuzzy spaces
- How to solve them using matrix model techniques
- Obtained "rigurous" definition of certain functional integrals

Can treat similarly:

- Scalar QFT on fuzzy $\mathbb{C}P^n$
- Scalar QFT on fuzzy $\mathbb{C}P^n \times \mathbb{R}$ (matrix quantum mechanics)

Future directions:

- Study regularization properties
- Deformed integrable hierarchies in these models?
- Relation to $c > 1$ string theories?

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