M-branes and Loop Spaces

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Based on:
- CS, CMP 305 (2011) 513
- S. Palmer and CS, JHEP 10 (2011) 008
- C. Papageorgakis and CS, JHEP 05 (2011) 099

also: work by Andreas Gustavsson

Motivation

There might be an effective descripion of M5-branes.

- Effective description of M2-branes proposed in 2007.
- This created lots of interest:
  BLG-model: >542 citations, ABJM-model: >740 citations

Question: Is there a similar description for M5-branes?

Possible way to approach the problem: Look at BPS subsector

- This was how the M2-brane models were derived originally.
- BPS subsector is interesting itself: Integrability

Results so far

- Integrability is also found here
- Using loop space is very helpful in this context
- Loop space appears naturally in proposed M5-brane equations

Integrability: Overview

Integrability in the D1-D3-brane system suggest analogous results for M2-M5-branes.

String theory picture:

- ∃ Twistor description of magnetic monopoles — Hitchin
- underlies Atiyah-Drinfeld-Hitchin-Manin-Nahm construction
- ADHMN is special variant of Fourier-Mukai or T-duality
- ⇒ ADHMN in terms of D1-D3-branes — Diaconescu, Tsimpis
- allows to switch perspective from D1- to D3-branes

M-theory picture:

- M-theory analogue of monopoles: self-dual strings
- ∃ Twistor description of SDS — M. Wolf & CS [1111.2539]

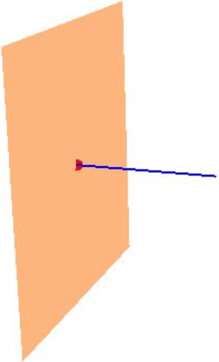
We therefore expect:

There is a generalized ADHMN construction that connects the effective descriptions of M2- and M5-brane BPS sectors.

D1-D3-Branes and the Nahm Equation

D1-branes ending on D3-branes can be described by the Nahm equation.

dim	0	1	2	3	...	6
$D1$	$\times$					$\times$
$D3$	$\times$	$\times$	$\times$	$\times$		



k D1-branes ending on D3-branes:

A Monopole appears.

$X^i \in \mathfrak{u}(k)$: transverse fluctuations

Nahm equation: $(s = x^6)$

$$\frac{\mathrm{d}}{\mathrm{d}s} X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

Note $\mathsf{SO}(3)$-invariance.

Solution: $X^i = r(s) G^i$ with

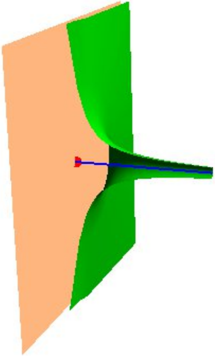
$$r(s) = \frac{1}{s}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

Nahm, Diaconescu, Tsimpis

D1-D3-Branes and the Nahm Equation

The D1-branes end on the D3-branes by forming a fuzzy funnel.

dim	0	1	2	3	…	6
$D1$	$\times$					$\times$
$D3$	$\times$	$\times$	$\times$	$\times$		



Solution: $X^i = r(s)G^i$

$$r(s) = \frac{1}{s}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

The D1-branes form a fuzzy funnel:

G^i form irrep of $\mathfrak{su}(2)$:
coordinates on fuzzy sphere S^2_F

D1-worldvolume polarizes: $2d \to 4d$

Myers

Lifting D1-D3-Branes to M2-M5-Branes

The lift to M-theory is performed by a T-duality and an M-theory lift

IIB	0	1	2	3	4	5	6
$D1$	×						×
$D3$	×	×	×	×			

T-dualize along x^5:

IIA	0	1	2	3	4	5	6
$D2$	×					×	×
$D4$	×	×	×	×		×	

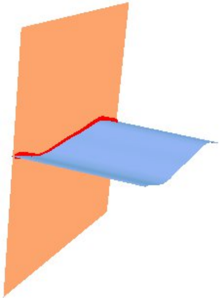
Interpret x^4 as M-theory direction:

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

The Basu-Harvey Lift of the Nahm Equation

M2-branes ending on M5-branes yield a Nahm equation with a cubic term.

M	0	1	2	3	4	5	6
$M2$	$\times$					$\times$	$\times$
$M5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	



Basu, Harvey, hep-th/0412310

A Self-Dual String appears.

Substitute SO(3)-inv. Nahm eqn.

$$\frac{\mathrm{d}}{\mathrm{d}s}X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

by the SO(4)-invariant equation

$$\frac{\mathrm{d}}{\mathrm{d}s}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

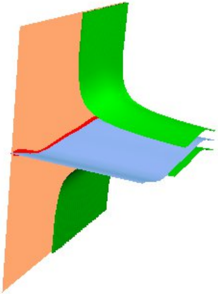
Solution: $X^\mu = r(s)G^\mu$ with

$$r(s) = \frac{1}{\sqrt{s}}\ , \quad G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

The Basu-Harvey Lift of the Nahm Equation

M2-branes ending on M5-branes yield a Nahm equation with a cubic term.

M	0	1	2	3	4	5	6
$M2$	$\times$					$\times$	$\times$
$M5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	



Solution: $X^\mu = r(s) G^\mu$

$$r(s) = \frac{1}{\sqrt{s}} \,, \quad G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

The M2-branes form a fuzzy funnel:

G^μ form a rep of $\mathfrak{so}(4)$:
coordinates on fuzzy sphere S^3_F

M2-worldvolume polarizes: $3d \to 6d$

- What is this triple bracket?
- What is a quantized S^3?

What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

BH: $$\frac{\mathrm{d}}{\mathrm{d}s}X^\mu + [A_s, X^\mu] + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0\ , \quad X^\mu \in \mathcal{A}$$

3-Lie algebra

Obviously: $\mathcal{A}$ is a vector space, $[\cdot,\cdot,\cdot]$ trilinear+antisymmetric.

Demand a "3-Jacobi identity," the fundamental identity:

$$[A,B,[C,D,E]] = [[A,B,C],D,E] + [C,[A,B,D],E] + [C,D,[A,B,E]]$$

Filippov (1985)

Gauge transformations from Lie algebra of inner derivations:

$$D : \mathcal{A} \wedge \mathcal{A} \to \mathrm{Der}(\mathcal{A}) =: \mathfrak{g}_\mathcal{A} \qquad D(A,B) \triangleright C := [A,B,C]$$

Commutator of inner dervs. closes due to fundamental identity.

Brief remarks on 3-Lie algebras

In analogy with Lie algebras, we can introduce 3-Lie algebras.

Examples:

Lie-algebras	3-Lie algebras
Heisenberg-algebra:	Nambu-Heisenberg 3-LA:
$[\tau_a, \tau_b] = \varepsilon_{ab} \mathbb{1}, \quad [\mathbb{1}, \cdot] = 0$	$[\tau_i, \tau_j, \tau_k] = \varepsilon_{ijk} \mathbb{1}, \quad [\mathbb{1}, \cdot, \cdot] = 0$
SU(2):	A_4:
$[\tau_i, \tau_j] = \varepsilon_{ijk} \tau_k$	$[\tau_\mu, \tau_\nu, \tau_\kappa] = \varepsilon_{\mu\nu\kappa\lambda} \tau_\lambda$

Note:

- Assume: Basu-Harvey equation is BPS equation of M2-brane model $\Rightarrow$ Bagger-Lambert-Gustavsson model
- Can endow 3-Lie algebras with metric compatible w. $\mathfrak{g}_{\mathcal{A}}$
- Only ones with pos. def. metric: $\oplus^k A_4$
- Problem: A_4 describes 2 M2-branes (?) How do we get more?

Real and Hermitian 3-Algebras

There are two natural generalizations of 3-Lie algebras.

Way out: sacrifice (manifest) SUSY

Real 3-Algebras ($\mathcal{N} = 2$)

Almost the same as 3-Lie algebras: triple bracket only antisymmetric in first two slots. S. Cherkis, CS, 0807.0808

Hermitian 3-Algebras ($\mathcal{N} = 6$)

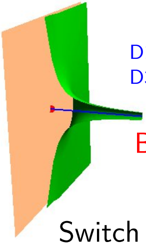
Start from a complex vector space $\mathcal{A}$. Bracket $[\,\cdot\,,\,\cdot\,;\,\cdot\,]$ satisfies

$$[A, B; C] = -[B, A; C], \;\; [\alpha A, B; C] := \alpha[A, B; C], \;\; [A, B; \alpha C] := \alpha^*[A, B; C]$$
$$[[C, D; E], A; B] - [[C, A; B], D; E] - [C, [D, A; B]; E] + [C, D; [E, B; A]] = 0$$

Bagger, Lambert, 0807.0163

Representation: $[A, B; C] := AC^\dagger B - BC^\dagger A$
Aharony, Bergman, Jafferis, Maldacena, 0806.1218

All our constructions generalize to these two types of 3-algebras.



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration!

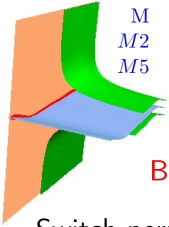
Switch perspective: D1→ D3:
Bogomolny monopole eqn.:

$$F_{ij} = \varepsilon_{ijk}\nabla_k\Phi \;\Rightarrow\; \nabla^2\Phi = 0$$

Single D3: Dirac monopole

$$\Phi = \frac{1}{r} \;\Rightarrow\; r(s) = \frac{1}{s}$$

$\Rightarrow$ matching profile!



M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

BPS configuration!

Switch perspective: M2→ M5:
Self-dual string eqn.:

$$H_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\partial_\sigma\Phi \;\Rightarrow\; \partial^2\Phi = 0$$

Only single M5 known:

$$\Phi = \frac{1}{r^2} \;\Rightarrow\; r(s) = \frac{1}{\sqrt{s}}$$

$\Rightarrow$ matching profile!

Dirac Monopoles and Principal U(1)-bundles

Dirac monopoles are described by principal U(1)-bundles over S^2.

Manifold M with cover $(U_i)_i$. Principal U(1)-bundle over M:

$$F \in \Omega^2(M, \mathfrak{u}(1)) ,$$
$$A_{(i)} \in \Omega^1(U_i, \mathfrak{u}(1)) \text{ with } F = \mathrm{d}A_{(i)}$$
$$g_{ij} \in \Omega^0(U_i \cap U_j, \mathsf{U}(1)) \text{ with } A_{(i)} - A_{(j)} = \mathrm{d}\log g_{ij}$$

Consider monopole in $\mathbb{R}^3$, but describe it on S^2 around monopole:

S^2 with patches U_+, U_-, $U_+ \cap U_- \sim S^1$: $g_{+-} = \mathrm{e}^{-\mathrm{i}k\phi},\ k \in \mathbb{Z}$

$$c_1 = \frac{\mathrm{i}}{2\pi} \int_{S^2} F = \frac{\mathrm{i}}{2\pi} \int_{S^1} A^+ - A^- = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, k = k$$

Monopole charge: k

Self-Dual Strings and Abelian Gerbes

Self-dual strings are described by abelian gerbes.

Manifold M with cover $(U_i)_i$. Abelian (local) gerbe over M:

$$H \in \Omega^3(M, \mathfrak{u}(1)) \ ,$$
$$B_{(i)} \in \Omega^2(U_i, \mathfrak{u}(1)) \text{ with } H = \mathrm{d}B_{(i)}$$
$$A_{(ij)} \in \Omega^1(U_i \cap U_j, \mathfrak{u}(1)) \text{ with } B_{(i)} - B_{(j)} = \mathrm{d}A_{ij}$$
$$h_{ijk} \in \Omega^0(U_i \cap U_j \cap U_k, \mathfrak{u}(1)) \text{ with } A_{(ij)} - A_{(ik)} + A_{(jk)} = \mathrm{d}h_{ijk}$$

Note: Local gerbe: principal $\mathsf{U}(1)$-bundles on intersections $U_i \cap U_j$.

Consider S^3, patches U_+, U_-, $U_+ \cap U_- \sim S^2$: bundle over S^2
Reflected in: $H^2(S^2, \mathbb{Z}) \cong H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$

$$\frac{\mathrm{i}}{2\pi} \int_{S^3} H = \frac{\mathrm{i}}{2\pi} \int_{S^2} B_+ - B_- = \ldots = k$$

Charge of self-dual string: k

Abelian Gerbes and Loop Space

By going to loop space, one can reduce differential forms by one degree.

Consider the following double fibration:

$$\begin{array}{ccc} & \mathcal{L}M \times S^1 & \\ {}^{ev}\swarrow & & \searrow^{pr} \\ M & & \mathcal{L}M \end{array}$$

Identify $T\mathcal{L}M = \mathcal{L}TM$, then: $x \in \mathcal{L}M \Rightarrow \dot{x}(\tau) \in T\mathcal{L}M$

Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \to \Omega^k(\mathcal{L}M)~, \quad v_i = \oint \mathrm{d}\tau\, v_i^\mu(\tau) \frac{\delta}{\delta x^\mu(\tau)} \in T\mathcal{L}M$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \oint_{S^1} \mathrm{d}\tau\, \omega(x(\tau))(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau))$$

An abelian local gerbe over M is a principal $\mathsf{U}(1)$-bundle over $\mathcal{L}M$.

$$\mathcal{T}\omega := \oint \mathrm{d}\tau \tfrac{1}{k!}\omega_{\mu_1\dots\mu_{k+1}}(x(\tau))\dot{x}^{\mu_{k+1}}(\tau)\delta x^{\mu_1}(\tau)\wedge\dots\wedge\delta x^{\mu_k}(\tau)\ .$$

Properties:

- Note that the integral is not only over coefficients!
- $\mathcal{T}$ is reparameterization invariant
- $\mathcal{T}$ is a chain map:

$$H = \mathrm{d}B \quad\Rightarrow\quad \mathcal{T}H = \delta\mathcal{T}B \qquad\qquad \mathrm{d}H = 0 \quad\Rightarrow\quad \delta\mathcal{T}H = 0$$

- $\mathcal{T}\mathrm{d}f = 0$ for all $f \in \mathcal{C}^\infty(M)$.
- $\mathcal{T}$ is not surjective!
- Inverse (“regression”) only on the image of $\mathcal{T}$.
- There are more line bundles over $\mathcal{L}M$ than gerbes on M

Side Remark: Quantization of S^3?

In the quantization problem, one is naturally led to loop space.

Geometric quantization prescription: (e.g. fuzzy sphere)

Special symplectic manifold (M,ω) → line bundle L with (h,∇) over M → Hilbert space $\mathscr{H}$: global holomorphic sections of L

Quantization map: $[\hat{f},\hat{g}] = \mathrm{i}\hbar\widehat{\{f,g\}} + \mathcal{O}(\hbar^2)$

Here: 2-plectic manifold (M,ϖ), $\varpi \in \Omega^3(M)$, hol. secs. of gerbe?

Quantization map: $[\hat{f},\hat{g},\hat{h}] = \mathrm{i}\hbar\widehat{\{f,g,h\}} + \mathcal{O}(\hbar^2)$?

Bracket on one-forms: $\{\alpha,\beta\} = \iota_{X_\alpha}\iota_{X_\beta}\varpi$, $\iota_{X_\alpha} = \mathrm{d}\alpha$ (Baez et al.)

Bracket antisymmetric, but Jacobi-identity violated by $\mathrm{d}\iota_{X_\alpha}\beta$.

Also: Quantization of one-forms?

Solution: ω on $\mathcal{L}M$ as $\omega := \mathcal{T}\varpi$. $\Rightarrow \mathcal{T}\{\alpha,\beta\}_\varpi = \{\mathcal{T}\alpha,\mathcal{T}\beta\}_{\mathcal{T}\varpi}$, then proceed as above

Brief Example: Quantization of $\mathbb{R}^3$

Loop space picture recovers formulas found in M-theory.

We start from $\mathbb{R}^3$ with 2-plectic form $\varpi = \varepsilon_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$.

Transgression yields a symplectic form on loop space $\mathcal{L}\mathbb{R}^3$:

$$\omega = \mathcal{T}\varpi = \oint \mathrm{d}\tau \oint \mathrm{d}\sigma\ \varepsilon_{ijk}\dot{x}^k(\tau)\delta(\tau-\sigma)\ \delta x^i(\tau) \wedge \delta x^j(\sigma)$$

Invert ω and obtain the Poisson bracket

$$\{f,g\} := \oint \mathrm{d}\tau\ \oint \mathrm{d}\sigma\, \delta(\tau-\sigma)\, \theta^{ijk}\, \frac{\dot{x}_k(\sigma)}{|\dot{x}(\sigma)|^2}\ \left(\frac{\delta}{\delta x^i(\tau)} f\right) \left(\frac{\delta}{\delta x^j(\sigma)} g\right)$$

or

$$[x^i(\tau), x^j(\sigma)] = \theta^{ijk}\frac{\dot{x}_k(\tau)}{|\dot{x}(\tau)|^2}\delta(\tau-\sigma) + \mathcal{O}(\theta^2)$$

This agrees with Kawamoto, Sasakura and Bergshoeff et al. (2000)

Quantum corrections: CS, Richard Szabo 1203.????

possibly more in: Richard's talk next week

Transgressed Self-Dual Strings

By going to loop space, one can reduce differential forms by one degree.

Recall the self-dual string equation on $\mathbb{R}^4$: $H_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\lambda} \frac{\partial}{\partial x^\lambda} \Phi$

Its transgressed form is an equation for a 2-form F on $\mathcal{L}\mathbb{R}^4$:

$$F(X,Y) = \oint \mathrm{d}\tau\, \varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\tau) \left. \frac{\partial}{\partial y^\lambda} \Phi(y) \right|_{y=x(\tau)} X^\mu(\tau) Y^\nu(\tau)$$

or

$$F_{(\mu\sigma)(\nu\rho)} = \delta(\sigma-\rho) \varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\tau) \left. \frac{\partial}{\partial y^\lambda} \Phi(y) \right|_{y=x(\tau)}$$

Extend to full non-abelian loop space curvature:

$$\begin{aligned} F^{\pm}_{(\mu\sigma)(\nu\tau)} = &\left(\varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\sigma) \nabla_{(\lambda\tau)} \Phi\right)_{(\sigma\tau)} \\ &\mp \left(\dot{x}_\mu(\sigma) \nabla_{(\nu\tau)} \Phi + \dot{x}_\nu(\sigma) \nabla_{(\mu\tau)} \Phi - \delta_{\mu\nu} \dot{x}^\kappa(\sigma) \nabla_{(\kappa\tau)} \Phi\right)_{[\sigma\tau]} \end{aligned}$$

where $\nabla_{(\mu\sigma)} := \oint \mathrm{d}\tau \delta x^\mu(\tau) \wedge \left(\frac{\delta}{\delta x^\mu(\tau)} + A_{(\mu\tau)} \right)$

Goal: Construct solutions to this equation.

The ADHMN Construction

There is a map between monopoles and solutions to the Nahm equations.

Nahm transform: Instantons on $T^4 \mapsto$ instantons on $(T^4)^*$

Roughly here:

$$T^4\colon \begin{cases} 3 \text{ rad. } 0 \\ 1 \text{ rad. } \infty \ : \ \text{D1 WV} \end{cases} \quad \text{and} \quad (T^4)^*\colon \begin{cases} 3 \text{ rad. } \infty \ : \ \text{D3 WV} \\ 1 \text{ rad. } 0 \end{cases}$$

Introduce (twisted) "Dirac operators":

$$\nabla\!\!\!/_{s,x} = -\mathbb{1}\tfrac{\mathrm{d}}{\mathrm{d}s} + \sigma^i \otimes (\mathrm{i}X^i + x^i \mathbb{1}_k) \ , \quad \bar{\nabla\!\!\!/}_{s,x} := \mathbb{1}\tfrac{\mathrm{d}}{\mathrm{d}s} + \sigma^i \otimes (\mathrm{i}X^i + x^i \mathbb{1}_k)$$

Properties:

$\Delta_{s,x} := \bar{\nabla\!\!\!/}_{s,x}\nabla\!\!\!/_{s,x} > 0$, $[\Delta_{s,x}, \sigma^i] = 0 \Leftrightarrow X^i$ satisfy Nahm eqn.

Normalized zero modes: $\bar{\nabla\!\!\!/}_{s,x}\psi_{s,x,\alpha} = 0$, $\mathbb{1} = \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}_{s,x}\psi_{s,x}$ yield:

$$A_\mu := \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}_{s,x} \frac{\partial}{\partial x^\mu} \psi_{s,x} \quad \text{and} \quad \Phi := -\mathrm{i} \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}_{s,x}\, s\, \psi_{s,x}$$

This is a solution to the Bogomolny monopole equations!

Examples: Dirac monopoles

One can easily construct Dirac monopole solutions using the ADHMN construction.

Charge 1: Nahm eqn: $\partial_s X^i = 0$, so put $X^i = 0$. Zero mode:

$$\psi_+ = \mathrm{e}^{-sR}\frac{\sqrt{R+x^3}}{x^1 - \mathrm{i}x^2}\begin{pmatrix} x^1 - \mathrm{i}x^2 \\ R - x^3 \end{pmatrix}$$

Monopole solution:

$$\Phi^+ = -\frac{\mathrm{i}}{2R}\ ,\ \ A_i^+ = \frac{\mathrm{i}}{2(x^1+x^2)^2}\left(x^2\left(1-\frac{x^3}{R}\right), -x^1\left(1-\frac{x^3}{R}\right), 0\right)$$

Charge 2: Nahm eqn. nontrivial. Choose:

$$X^i = -\frac{1}{s}T^i \quad \text{with} \quad T^i = \frac{\sigma^i}{2\mathrm{i}} = -\bar{T}^i$$

Resulting solution:

$$\Phi^+ = -\frac{\mathrm{i}}{R}\ ,\ \ A_i^+ = \dots$$

Lift of the "Dirac operator"

There is a natural lift of the Dirac operator to M-theory.

Type IIB (twisted):

$$\nabla\!\!\!\!/^{\mathrm{IIB}}_{s,x} = -\mathbb{1}\frac{\mathrm{d}}{\mathrm{d}s} + \sigma^i(\mathrm{i}X^i + x^i\mathbb{1}_k)$$

IIB	0	1	2	3	4	5	6
$D1$	×						×
$D3$	×	×	×	×			

Type IIA (twisted):

$$\nabla\!\!\!\!/^{\mathrm{IIA}}_{s,x} = -\gamma_5\mathbb{1}_k\frac{\mathrm{d}}{\mathrm{d}s} + \gamma^4\gamma^i(X^i - \mathrm{i}x^i)$$

IIA	0	1	2	3	4	5	6
$D2$	×					×	×
$D4$	×	×	×	×		×	

M-theory (untwisted):

$$\nabla\!\!\!\!/^{\mathrm{M}}_{s} = -\gamma_5\frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}D(X^\mu, X^\nu)$$

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

M-theory (twisted):

$$\nabla\!\!\!\!/^{\mathrm{M}}_{s,x(\tau)} = -\gamma_5\frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}\left(D(X^\mu, X^\nu) - \mathrm{i}\oint \mathrm{d}\tau\, x^\mu(\tau)\dot{x}^\nu(\tau)\right)$$

Lifted ADHMN Construction

The lifted ADHMN construction yields solutions to the loop space self-dual string eqns.

Recall: $\Delta^{\rm IIB} := \bar{\nabla\!\!\!/}^{\rm IIB}\nabla\!\!\!/^{\rm IIB}$, $[\Delta^{\rm IIB}, \sigma^i] = 0 \Leftrightarrow X^i$ satisfy Nahm eqn.

Here: $\Delta^{\rm M} := \bar{\nabla\!\!\!/}^{\rm M}\nabla\!\!\!/^{\rm M}$, $[\Delta, \gamma^{\mu\nu}] = 0 \Leftarrow X^\mu$ satisfy BH eqn.

Recall extended self-dual string equation on loop space:

$$F^{\pm}_{(\mu\sigma)(\nu\tau)} = \big(\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi\big)_{(\sigma\tau)} \mp \big(\dot{x}_\mu(\sigma)\nabla_{(\nu\tau)}\Phi + \dot{x}_\nu(\sigma)\nabla_{(\mu\tau)}\Phi - \delta_{\mu\nu}\dot{x}^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\big)_{[\sigma\tau]}$$

From normalized, $\mathcal{A}$-valued zero modes $\psi_{s,x(\tau)}$ of $\bar{\nabla\!\!\!/}^{\rm M}$ construct

$$A_{(\mu\tau)} = \int \mathrm{d}s\, \bar{\psi}_{s,x} \frac{\delta}{\delta x^\mu(\tau)} \psi_{s,x}\ , \quad \Phi = -\mathrm{i} \int \mathrm{d}s\, \bar{\psi}_{s,x}\, s\, \psi_{s,x}$$

These fields solve the loop space self-dual string equation.

Verification of the Construction

Verifying the construction is rather straightforward.

The proof is easy and follows that of the ADHMN construction:

$$
\begin{aligned}
F^{ab}_{(\mu\sigma)(\nu\tau)} &\overset{[\cdot]}{=} 2\int_{\mathcal{I}} \mathrm{d}s\, (\delta_{(\mu\sigma)}\bar\psi^a_{s,x}, \delta_{(\nu\tau)}\psi^b_{s,x}) + 2\int_{\mathcal{I}} \mathrm{d}s \int_{\mathcal{I}} \mathrm{d}t\, (\bar\psi^a_{s,x}, \delta_{(\mu\sigma)}\psi^c_{s,x})(\bar\psi^c_{t,x}, \delta_{(\nu\tau)}\psi^b_{t,x}) \\
&\overset{[\cdot]}{=} -2\int_{\mathcal{I}} \mathrm{d}s \int_{\mathcal{I}} \mathrm{d}t \left(\delta_{(\mu\sigma)}\bar\psi^a_{s,x}\,,\ (\not{\nabla}_{s,x} G_x(s,t) \bar{\not{\nabla}}_{t,x})\, \delta_{(\nu\tau)}\psi^b_{t,x}\right) \\
&\overset{[\cdot]}{=} 2\int_{\mathcal{I}} \mathrm{d}s \int_{\mathcal{I}} \mathrm{d}t \left(\bar\psi^a_{s,x}, \left(\gamma^{\mu\kappa}\dot x^\kappa(\sigma) G_x(s,t)\gamma^{\nu\lambda}\dot x^\lambda(\tau)\right) \psi^b_{t,x}\right) \\
&\overset{[\cdot]}{=} 2\varepsilon_{\mu\nu\kappa\lambda}\int_{\mathcal{I}} \mathrm{d}s \int_{\mathcal{I}} \mathrm{d}t \left(\bar\psi^a_{s,x},\, G_x(s,t)\gamma^{\kappa\rho}\gamma_5 \dot x^\lambda(\sigma)\dot x^\rho(\tau)\psi^b_{t,x}\right) \\
&\qquad + \int_{\mathcal{I}} \mathrm{d}s \int_{\mathcal{I}} \mathrm{d}t \left(\bar\psi^a_{s,x},\, G_x(s,t) \left(4\gamma^{\mu\lambda}\dot x^\nu(\sigma)\dot x^\lambda(\tau) - 2\delta^{\mu\nu}\gamma^{\kappa\lambda}\dot x^\kappa(\sigma)\dot x^\lambda(\tau)\right) \psi^b_{t,x}\right) \\
&\overset{[\cdot]}{=} \mathrm{i}\varepsilon_{\mu\nu\kappa\lambda}\dot x^\kappa(\sigma)\int_{\mathcal{I}} \mathrm{d}s \left((\nabla_{(\lambda\tau)}\bar\psi_{s,x})^a,\ s\,\psi^b_{s,x}\right) + \left(\bar\psi^a_{s,x},\ s\,(\nabla_{(\lambda\tau)}\psi_{s,x})^b\right) \\
&\qquad \mp 2\mathrm{i}\dot x_\mu(\sigma)\int_{\mathcal{I}} \mathrm{d}s \left((\nabla_{(\nu\tau)}\bar\psi_{s,x})^a,\ s\,\psi^b_{s,x}\right) + \left(\bar\psi^a_{s,x},\ s\,(\nabla_{(\nu\tau)}\psi_{s,x})^b\right) \\
&\qquad \mp 2\mathrm{i}\dot x_\nu(\sigma)\int_{\mathcal{I}} \mathrm{d}s \left((\nabla_{(\mu\tau)}\bar\psi_{s,x})^a,\ s\,\psi^b_{s,x}\right) + \left(\bar\psi^a_{s,x},\ s\,(\nabla_{(\mu\tau)}\psi_{s,x})^b\right) \\
&\qquad \pm \mathrm{i}\delta_{\mu\nu}\dot x^\kappa(\sigma)\int_{\mathcal{I}} \mathrm{d}s \left((\nabla_{(\kappa\tau)}\bar\psi_{s,x})^a,\ s\,\psi^b_{s,x}\right) + \left(\bar\psi^a_{s,x},\ s\,(\nabla_{(\kappa\tau)}\psi_{s,x})^b\right) \\
&\overset{[\cdot]}{=} \left(\varepsilon_{\mu\nu\kappa\lambda}\dot x^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi \mp \dot x_\mu(\sigma)\nabla_{(\nu\tau)}\Phi \mp \dot x_\nu(\sigma)\nabla_{(\mu\tau)}\Phi \pm \delta_{\mu\nu}\dot x^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\right)^{ab}\,.
\end{aligned}
$$

Reduction to the ADHMN Construction

The lift reduces in the expected way to the ADHMN construction.

On $\mathcal{L}S^3 \subset \mathcal{L}\mathbb{R}^4$: $x^\mu x^\mu = \dot{x}^\mu \dot{x}^\mu = R^2$, $x^\mu \dot{x}^\mu = 0$.

Reduction (cf. Mukhi, Papageorgakis, 0803.3218):

$$\langle X^4 \rangle = \frac{r}{\ell_p^{3/2}} e_4 = g_{\mathrm{YM}} e_4 \ , \quad \dot{x}^4(\tau) = R \Rightarrow \dot{x}^i(\tau) = x^4(\tau) = 0$$

$$\nabla\!\!\!\!/^{\mathrm{M}} = -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}\left(D(X^\mu, X^\nu) - \mathrm{i} \oint \mathrm{d}\tau x^\mu(\tau)\dot{x}^\nu(\tau) \right)$$

$$\to \ -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}\left(D(X^\mu, X^\nu) - 2\pi \mathrm{i} R x_0^\mu \delta_4^\nu \right)$$

$$= -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + R\gamma^{4i}\left(X^{i\alpha} D(e_\alpha, e_4) - \mathrm{i} x_0^i \right) + \ldots = \nabla\!\!\!\!/^{\mathrm{IIA}} + \ldots$$

$$\frac{\mathrm{d}}{\mathrm{d}s} X^\mu = \tfrac{1}{3!}\varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] \quad \to \quad \frac{\mathrm{d}}{\mathrm{d}s} X^i = \tfrac{1}{2}\varepsilon^{ijk} R [X^j, X^k] + \ldots$$

$$F_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} \dot{x}^\rho \frac{\partial}{\partial x^\sigma} \Phi_{\mathrm{SDS}} \quad \to \quad F_{ij} = \varepsilon_{ijk} \frac{\partial}{\partial x^k} R \Phi_{\mathrm{SDS}} + \ldots$$

Charge 1: Choose again trivial Nahm data. Introduce

$$y^{\mu\nu} := \oint \mathrm{d}\tau\, x^{[\mu}(\tau)\dot{x}^{\nu]}(\tau)\ , \quad r_\pm^2 := \tfrac{1}{2}\sqrt{(y^{\mu\nu} \pm \tfrac{1}{2}\varepsilon_{\mu\nu\kappa\lambda}y^{\kappa\lambda})^2}$$

The zero modes of the adjoint of the Dirac operators are:

$$\psi_{s,x} \sim \mathrm{e}^{-r_-^2 s}\begin{pmatrix} \mathrm{i}\left(r_-^2 + y^{12} - y^{34}\right) \\ y^{13} + y^{24} + \mathrm{i}(y^{23} - y^{14}) \\ 0 \\ 0 \end{pmatrix}$$

The solution then reads as $\Phi = \frac{\mathrm{i}}{2r_-^2}$ and

$$A(\sigma) = \frac{\mathrm{i}}{2r_-^2\left(r_-^2 + (y^{12} - y^{34})\right)}\begin{pmatrix} \dot{x}^3(\sigma)(y^{23} - y^{14}) + \dot{x}^4(\sigma)(y^{13} + y^{24}) \\ \dot{x}^4(\sigma)(y^{23} - y^{14}) - \dot{x}^3(\sigma)(y^{13} + y^{24}) \\ \dot{x}^1(\sigma)(y^{14} - y^{23}) + \dot{x}^2(\sigma)(y^{13} + y^{24}) \\ \dot{x}^2(\sigma)(y^{14} - y^{23}) - \dot{x}^1(\sigma)(y^{13} + y^{24}) \end{pmatrix}$$

This is indeed a solution.

Charge 2:

Nahm data:

$$X^\mu = \frac{e_\mu}{\sqrt{2s}}\ , \quad e_\mu \text{ generate } A_4$$

Solution:

$$\Phi(x) = \frac{\mathrm{i}}{r_-^2}$$

As expected: twice the charge of the case $k = 1$.

Examples: Arbitrary charge k

For arbitrary topological charge, one has to switch to the ABJM model.

BLG model with A_4 can only describe $k=2$ M2-branes. $\Rightarrow$ ABJM:

M	0	1	2	3	4	5	6
$M2$	$\times$					$\times$	$\times$
$M5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	

Recall: BLG $\rightarrow$ ABJM by $\mathsf{SO}(8)\rightarrow\mathsf{SU}(4)$. Here: $\mathsf{SO}(4)\rightarrow\mathsf{SU}(2)$

Basu-Harvey equation for the ABJM model (various people, 2008):

$$\frac{\mathrm{d}}{\mathrm{d}s}Z^\alpha=\tfrac{1}{2}(Z^\alpha\bar{Z}_\beta Z^\beta-Z^\beta\bar{Z}_\beta Z^\alpha)~,\quad \alpha,\beta=1,2~.$$

Irreducible solutions have $Z^1=\bar{Z}_1$, only three components!
Nevertheless: construct Dirac operator, zero modes etc. and find:

$$\Phi=\frac{\mathrm{i}k}{2r^2}\,\mathbb{1}_2~.$$

Remarks

Our lift of the ADHMN construction is very natural and rather straightforward.

- The lift of the Dirac operator was natural considering the corresponding brane configurations.
- It is natural to go to loop space to describe self-dual strings.
- It can be trivially rendered nonabelian:
- The construction nicely involves the Basu-Harvey equation.
- It reduces naturally to the ADHMN construction.
- The construction does produce transgressed self-dual strings.
- It extends to real and hermitian 3-algebras ($\rightarrow$ ABJM model)
- In these constructions: Gauge group on loop space is $G \times G$. cf. M5-brane models, e.g. Chu [1108.5131]
- Fits with expectations from twistor space picture.

CS, M. Wolf [1111.2539]

Recall the transgression map:

$$\mathcal{T}\omega := \oint \mathrm{d}\tau \tfrac{1}{k!}\omega_{\mu_1\dots\mu_{k+1}}(x(\tau))\dot{x}^{\mu_{k+1}}(\tau)\delta x^{\mu_1}(\tau)\wedge\dots\wedge\delta x^{\mu_k}(\tau)\ .$$

Equations found by Lambert, Papageorgakis, 1007.2982:

$$\nabla^2 X^I - \tfrac{\mathrm{i}}{2}[\bar{\Psi},\Gamma_\nu\Gamma^I\Psi,C^\nu] - [X^J,C^\nu,[X^J,C_\nu,X^I]] = 0$$

$$\Gamma^\mu\nabla_\mu\Psi - [X^I,C^\nu,\Gamma_\nu\Gamma^I\Psi] = 0$$

$$\nabla_{[\mu}H_{\nu\lambda\rho]} + \tfrac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[X^I,\nabla^\tau X^I,C^\sigma] + \tfrac{\mathrm{i}}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[\bar{\Psi},\Gamma^\tau\Psi,C^\sigma] = 0$$

$$F_{\mu\nu} - D(C^\lambda,H_{\mu\nu\lambda}) = 0$$

$$\nabla_\mu C^\nu = D(C^\mu,C^\nu) = 0$$

$$D(C^\rho,\nabla_\rho X^I) = D(C^\rho,\nabla_\rho\Psi) = D(C^\rho,\nabla_\rho H_{\mu\nu\lambda}) = 0$$

Factorization of $C^\rho = C\dot{x}^\rho$. Here, 3-Lie algebra transgression:

$$(\mathcal{T}\omega)_x(v_1(\tau),\dots,v_k(\tau)) := \int_{S^1}\mathrm{d}\tau\, D(\omega(v_1(\tau),\dots,v_k(\tau),\dot{x}(\tau)),C)$$

The non-abelian tensor multiplet on loop space

The corresponding equations can all be rewritten on loop space.

Transgression of matter fields (not in Huang, Huang, 1008.3834)

$$\mathring{X}^I(x(\tau)) := R\,D(C, X^I(x(\tau)))\ ,\quad \mathring{\Psi}(x(\tau)) := \Gamma^\rho \dot{x}_\rho D(C, \Psi(x(\tau)))$$

Equations of motion (SYM-like):

$$\nabla^2 \mathring{X}^I + \tfrac{\mathrm{i}}{2}\tfrac{1}{R}\dot{x}^\nu[\bar{\mathring{\Psi}}, \Gamma_\nu \Gamma^I \mathring{\Psi}] + [\mathring{X}^J, [\mathring{X}^J, \mathring{X}^I]] = 0\ ,$$

$$\tfrac{1}{R}\Gamma^{\mu\nu}\dot{x}_\nu \nabla_\mu \mathring{\Psi} - \Gamma^I[\mathring{X}^I, \mathring{\Psi}] = 0\ ,$$

$$\nabla_\mu \mathring{F}^{\mu\nu} + [\mathring{X}^I, \nabla^\nu \mathring{X}^I] + \tfrac{\mathrm{i}}{2}\left([\bar{\mathring{\Psi}}, \Gamma^\nu \mathring{\Psi}] - \tfrac{2}{R^2}\dot{x}^\sigma \dot{x}^\nu [\bar{\mathring{\Psi}}, \Gamma_\sigma \mathring{\Psi}]\right) = 0\ ,$$

Supersymmetry transformations (SYM-like):

$$\begin{aligned}
\delta \mathring{X}^I &= \tfrac{1}{R}\mathrm{i}\bar{\varepsilon}\Gamma^I \dot{x}^\rho \Gamma_\rho \mathring{\Psi}\ ,\\
\delta \mathring{A}_\mu &= \tfrac{1}{R^2}\mathrm{i}\bar{\varepsilon}\Gamma_{\mu\lambda}\Gamma_\rho \dot{x}^\lambda \dot{x}^\rho \mathring{\Psi}\ ,\\
\delta \mathring{\Psi} &= \tfrac{1}{R}\Gamma^{\nu\mu}\dot{x}_\nu \Gamma^I \nabla_\mu \mathring{X}^I \varepsilon + \tfrac{1}{2}\Gamma_{\mu\nu}\mathring{F}^{\mu\nu}\varepsilon - \tfrac{1}{2}\Gamma^{IJ}[\mathring{X}^I, \mathring{X}^J]\varepsilon\ ,\\
\delta C^\mu &= 0
\end{aligned}$$

Remarks

The loop space tensor multiplet fits well into the picture.

- Get SYM theory on loop space from the tensor multiplet
- BPS equation is essentially the self-dual string equation on loop space:
$$F_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}\dot{x}^\rho \nabla^\sigma X^6$$
- We used a reduced loop space derivative: $\partial_\mu := \oint \mathrm{d}\tau \frac{\delta}{\delta x^\mu(\tau)}$
- Technical issues: Laplace operator on loop space etc.
- Right direction, more work in progress to get rid of $\dot{x}$ etc.

Summary:

- ✓ Reformulation of self-dual string equation on loop space
- ✓ Generalized ADHMN construction for self-dual string
- ✓ Classical integrability from twistor picture
- ✓ Explicit constructions in many cases
- ✓ Reformulate non-abelian tensor multiplet eqns. on loop space
- ✓ Partially generalized ADHMN construction

Future directions:

- ▷ Find (2,0)-theory using loop space
- ▷ Study classical integrability in more detail (spectral curves, ...)
- ▷ Higher analogues of Magnetic Bags
- ▷ Loop space quantization of S^3, cf. Dirac operator
- ▷ Quantization of S^3 via gerbes and 2-groupoids

M-branes and Loop Spaces

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