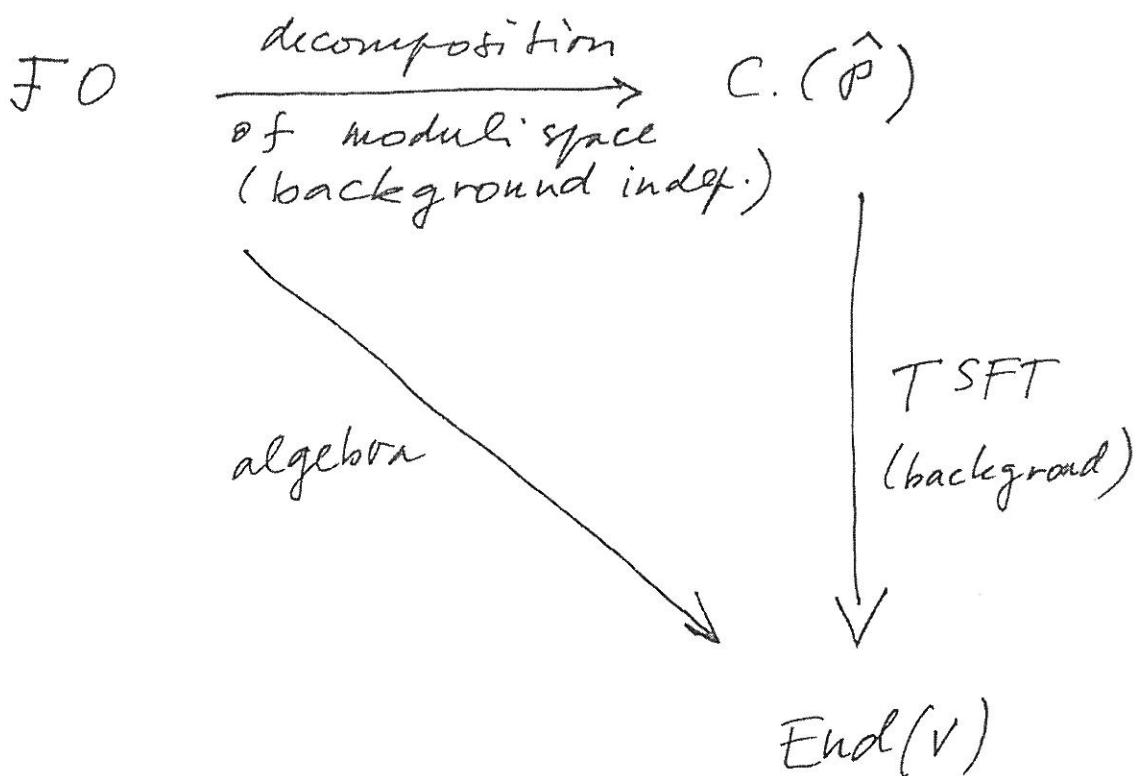


OPERADS, HOMOTOPY ALGEBRAS
AND STRINGS

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(6)

Commutative triangle
of morphisms of modular
operads



B.J. & K.M. Type II SFT: Geom. Approach
and operadic description
arXiv 1303.2323

M.D., B.J., K.M. Modular operads
and QOCMA
arXiv 1308.3223

Category of corollas

$$\left(\begin{array}{c} \times \\ G \\ \times \end{array} \right)_n \xrightarrow{\quad g \in \Sigma_n \quad} \textcircled{1}$$

$$2(G-1) + n > 0$$

operations

$$a \circ b \quad \begin{array}{c} \nearrow \\ \times \\ \searrow \end{array} \quad \begin{array}{c} a \\ b \\ \swarrow \\ \times \\ \nearrow \end{array} \quad \begin{array}{c} \leftrightarrow \\ (\Rightarrow \Leftarrow) \end{array} \quad \begin{array}{c} \times \\ \times \\ \times \end{array} \quad (m_1+1, G_1) \quad (m_2+1, G_2) \quad (m_1+m_2, G_1+G_2)$$

$$\xi_{ab} \quad \begin{array}{c} \nearrow \\ \times \\ \searrow \end{array} \quad \begin{array}{c} \leftrightarrow \\ (\Rightarrow \Leftarrow) \end{array} \quad \begin{array}{c} \rightarrow \\ \times \\ \times \end{array} \quad (m-2, G+1) \quad (m, G)$$

Modular operads

$$\mathcal{P}\left(\begin{array}{c} \times \\ G \\ \times \end{array} \right) \quad \text{collection of d.g. v.s. +}$$

degree 0 morphisms

$$\rho(g) : \mathcal{P}(\times) \rightarrow \mathcal{P}(g(\times))$$

$$a \circ b : \mathcal{P}\left(\begin{array}{c} \times \\ a \\ \times \end{array} \right) \otimes \mathcal{P}\left(\begin{array}{c} \times \\ b \\ \times \end{array} \right) \rightarrow \mathcal{P}(\times)$$

$$\xi_{ab} : \mathcal{P}\left(\begin{array}{c} \times \\ a \\ b \end{array} \right) \rightarrow \mathcal{P}(\times)$$

(2)

s.t. :

$$1. \quad a^{\circ} b (x \otimes y) = (-1)^{|x||y|} b^{\circ} a (y \otimes x)$$

$$2. \quad P(1_{\star}) = 1_{P(\star)}$$

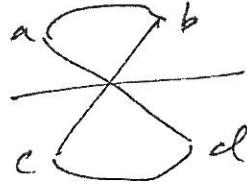
$$P(\rho\sigma) = P(\rho)P(\sigma)$$

$$3. \quad P(\rho|> \sqcup \sigma|<)_{a^{\circ} b} =$$

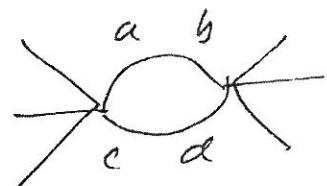
$$= \rho(a)^{\circ} \sigma(b) \quad P(\rho)P(\sigma)$$

$$4. \quad P(\rho|>) \xi_{ab} = \xi_{\rho(a)\rho(b)} P(\rho)$$

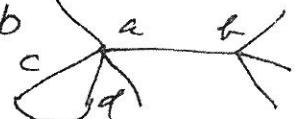
$$5. \quad \xi_{ab} \xi_{cd} = \xi_{cd} \xi_{ab}$$



$$6. \quad \xi_{ab} c^{\circ} d = \xi_{cd} a^{\circ} b$$

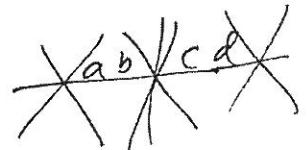


$$7. \quad (a^{\circ} b)(\xi_{cd} \otimes 1) = \xi_{cd} a^{\circ} b$$



$$8. \quad (a^{\circ} b)(1 \otimes c^{\circ} d) =$$

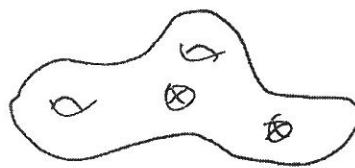
$$= c^{\circ} a (a^{\circ} b \otimes 1)$$



$\mathcal{G} = \mathcal{O} + \text{forgetting } \xi_{ab} \rightarrow \text{cyclic operad}$

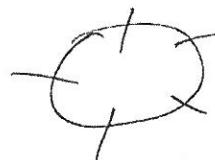
(3)

Ex.

① $\text{Mod}(\text{Com})$ 

$$\begin{array}{ccc} \cancel{\times}^n & \rightarrow & 1\text{-dim v.s. } \langle \cancel{\times}_a^n \rangle \end{array}$$

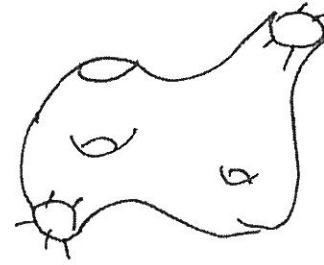
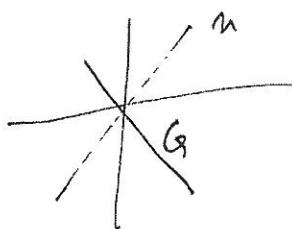
Σ_n action trivial $a^\circ b \models \text{lab orbits}$

② $\overset{c}{\text{Ass}}$ 

$$\begin{array}{ccc} \cancel{\times}^n & \rightarrow & \langle \text{cyclic orderings of } \cancel{\times}^n \rangle \end{array}$$

$$\Sigma_n \quad ((x_1, \dots, x_n)) \mapsto ((x_{\phi^{-1}(1)}, \dots, x_{\phi^{-1}(n)}))$$

$$a^\circ b : ((a, x_1, \dots, x_n)) \otimes ((b, y_1, \dots, y_n)) \mapsto \\ ((x_1, \dots, x_n, y_1, \dots, y_n))$$

③ $\text{Mod}(\overset{c}{\text{Ass}})$ 

$$\rightarrow \left\langle \begin{array}{c} \text{blob with } m_1 \text{ and } m_2 \text{ parts} \\ + \text{cyclic orderings on parts} \end{array} \right\rangle$$

$$((x_1, \dots, x_{m_1}))((x_{m_1+1}, \dots, x_{m_1+m_2}), \dots)$$

Σ_n action by permutations

$a^{\circ} b$

(3)

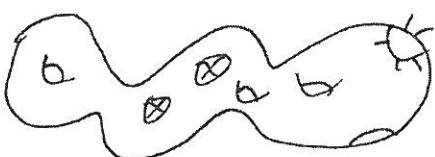
$$\begin{aligned}
 & ((\textcircled{1}) \dots (a, x_1, \dots, x_n)) \dots ((\textcircled{k})) a^{\circ} b \\
 & \quad ((\textcircled{(k+1)})) \dots ((b, y_1, \dots, y_m)) \dots \\
 = & ((x_1, \dots, x_n, y_1, \dots, y_m) (\textcircled{1}) \dots (\textcircled{k})) \dots
 \end{aligned}$$

ξ_{ab}

- 1) $((\textcircled{1}) \dots (a, x_1, \dots, x_k, b, x_{k+1}, \dots, x_n)) \dots (())$
 $\mapsto ((x_1, \dots, x_k)(x_{k+1}, \dots, x_n)) \dots ((\textcircled{1})) \dots$
- 2) $((\textcircled{1}) \dots (a, x_1, \dots, x_n)) \dots ((b, y_1, \dots, y_m)), ((\textcircled{1})) \dots$

④ Combination of
 $\text{Mod}(C^c)$ and $\text{Mod}(A^{ss})$

"2-coloured" (open & closed)
 ROC - operad



n -closed
 m -open $\sum_n \times \sum_m$
 action

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(5) Mod $(\lim_{N \rightarrow \infty} \text{Mod})$

your favorite
super Riemann
surface

4-colours

NS-NS

NS-R

R-NS

R-R

$$n_{NS-NS} + n_{NS-R} \in \mathbb{N}_0$$

$$n_{NS-NS} + n_{R-NS} \in \mathbb{N}_0$$

$$n_{R-R} + n_{R-NS} \in 2\mathbb{N}_0$$

$$n_{R-R} + n_{NS-R} \in 2\mathbb{N}_0$$

$$\sum_{NS-NS} \times \sum_{NS-R} \times \dots$$

action

$$a^0 b$$

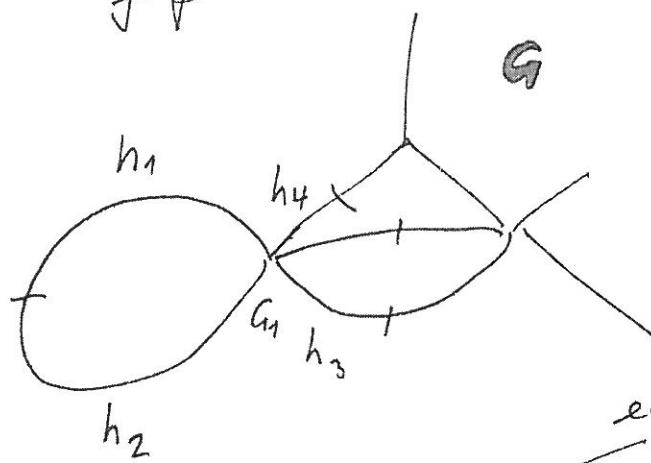
$$\xi_{ab}$$

non zero only for a, b of the same color

(6)

Feynman transform

FP



$$2(G_i - 1) + |V_i| > 0$$

Decorate G by $(\uparrow e_1 \wedge \dots \wedge \uparrow e_5) \otimes (\rho_1 \otimes \rho_2 \otimes \rho_3)$

$$\rho_1 \in P \left(\begin{array}{c} h_1 \quad h_5 \\ \cancel{a_1} \quad \cancel{h_4} \\ h_2 \quad h_3 \end{array} \right)^{\#}$$

$(a^o b)_{FP}$ $(\xi_{ab})_{FP}$ grafting of graphs
and attaching edges, resp

∂_{FP} adding edge +
modifications of decorations

using $(a^o b)^{\#}$ and $(\xi_{ab})^{\#}$

$$\partial_{FP} \cancel{*} = \sum_{(n, g)} \cancel{*} + *$$

$n_1 + n_2 = n$
 $g_1 + g_2 = g$

$G=0$ P-cyclic BAR CONSTRUCTION

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k -twisted modular operad

$a^b \otimes_{ab}$ are degree $\frac{1}{2}$

5. 6. 7. 8. get modified by signs

$$\xi_{ab} \xi_{cd} = - \xi_{cd} \xi_{ab}$$

$$\xi_{ab} \otimes^c d = - \xi_{cd} a^b$$

$$(a^b)(\xi_{cd} \otimes 1) = - \xi_{cd} a^b$$

$$(a^b)(1 \otimes c^d) = - c^d (a^b \otimes 1)$$

FP is k -twisted !!!

Ex $\text{End}(V)$ V d. g. (super) V.S.

ω - degree -1 symplectic

$$\omega(d \otimes 1) + \omega(1 \otimes d) = 0$$

$$P(n, g) = \text{Hom}(V^{\otimes n}, \mathbb{C}) =: E_V(n, g)$$

Σ_n - permutations of inputs

a^b , ξ_{ab} contracting inputs
using ω

(worse versions)

STRONG THEORY

$$d \leftrightarrow Q$$

$V \leftrightarrow$ CFT state space
 $W \hookrightarrow$ BPZ + insertions (sub)

Algebra over Feynman transf. 8

Morphism of twisted mod. operads

$$FP \longrightarrow \text{End } V$$

- FP algebra structure on V

Borannikov:

Algebra over $FP \Leftrightarrow$ following data

$$\left\{ \underbrace{m(c,g)}_{d} \in (\rho(c,g) \otimes \varepsilon_v(g,g))^{inv} \right\}$$

$$(d_{\varepsilon_A} - d_p)m(c,g) =$$

$$(\xi_{ab})_p \otimes (\xi_{av})_{\varepsilon_A} m(c \cup \{a,b\}, g^{-1})$$

$$+ \frac{1}{2} \sum_{c_1 \cup c_2 = c} ((a^{\circ b})_p \otimes (a^{\circ b})_{\varepsilon_A})$$

$$g_1 + g_2 = g$$

$$\tau(m(c_1 \cup \{a\}, g_1) \otimes m(c_2 \cup \{b\}, g_2))$$

$$(P, d, \Delta, \{, \}, \{, \}) \quad - \text{noncommutative } BV\text{-algebra on } V$$

$$(P, d + \Delta, \{, \}, \{, \}) \quad \text{d.g. (super) Lie algebra}$$

$$S := \sum_{\substack{n \geq 0 \\ g \geq 0}} \hbar^g m_{n,g}$$

Solution to the BV master eq.
quantum

$$dS + \hbar \Delta S + \frac{1}{2} \{ S, S \} = 0$$

Physics?

$$\left(\Phi(n, g) \otimes \sum_V (n, g) \right)^{\sum_n} \simeq \left(\Phi(n, g) \otimes V^{\# \otimes n} \right)^{\sum_n}$$

$$\sum p_i \otimes \psi^i \mapsto \frac{1}{n!} \sum \sum \psi^i(a_I) (p_i \otimes \sum \phi^I)$$

a_i -basis of V

$$a_I = a_{i_1} \otimes \dots \otimes a_{i_m}$$

$$\phi^I = \phi^{i_1} \otimes \dots \otimes \phi^{i_m}$$

(string fields)

S — BV action

$\delta, \{, \}$ — are really BV-operadins
on (non-commutative) fields ϕ^i

Ex. ① $\text{Mod}(\overset{\circ}{\mathcal{C}_m})$

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$$(\mathcal{P}(n, g) \otimes V^{\# \otimes n})_{\sum_m} \cong S^m(V^{\#})$$

$$S \in S(V^{\#})[[\hbar]]$$

$$S = \sum_{n, g} \frac{\hbar^g}{n!} \sum_I f_n^g(a_I) \phi^I$$

(graded symmetric)

Solutions to BV equations ($g=0, \overset{\circ}{\mathcal{L}_0}$)

\leftrightarrow boundary long algebras

Merkel, Zwierlein

② A_{∞}^c

$$S = \sum_m \frac{1}{m} \underbrace{f_m(a_I)}_{\text{cyclic symmetric}} \phi^I$$

A_{∞}^c Stasheff, Zwierlein

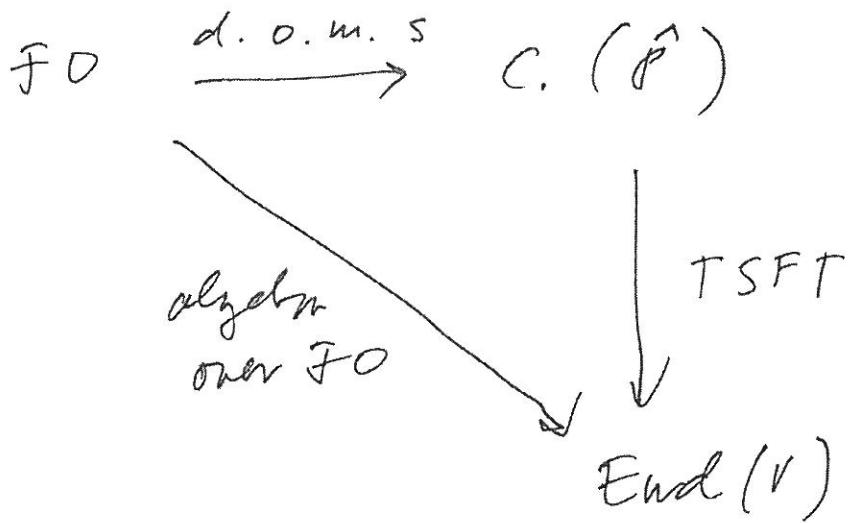
③ $\text{Mod}(A_{\infty}^c)$

Quantum A_{∞}^c $g = 2g + b - 1$

$$S = \sum_{g, b, m} \frac{\hbar^{g_1}}{b!} \frac{1}{m_1 \dots m_b} f^{\partial b} \underbrace{(a_I)}_{\text{cyclic}} \phi^{I_1} \dots \phi^{I_m}$$

(4) ROC special
S - Friedlands ROCSTF action
Q OCHA Minster & Sadiq
 $G=0$ Kajiwara & Stashoff

(5) Mod ($C_m^{e^N=1}$)
Type II super SFT



$C.(\hat{P})$ - a delicate, ⁱⁿ modular and twisted

- $F0 \xrightarrow{\text{morphism}} C.(\hat{P})$
- BV-structure on $(F0 \otimes C.(\hat{P}))$
morphism \Leftrightarrow solutions of
BV ME

- d.o.m.s provides for a (unique) solution (geometric vertices)
(constant prescription w.r.t. m vertices)
- TSFT - morphism of BV algebras gives algebraic vertices
 $(TSFT \circ \text{d.o.m.s}) = \text{constant. fn}$ of a SFT