# 2-Group Symmetries on Moduli Spaces in Higher Gauge Theory

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Why look at higher symmetries of moduli spaces?

- Quantization of a physical theory is a kind of linearization
- Classical symmetries of moduli spaces act on Hilbert spaces
- Gives representation theory of groupoids for extended field theories (e.g. WZW model as boundary condition of Chern-Simons, etc.)
- In higher gauge theory, "symmetry" is encoded in higher groupoids
- Higher quantization will involve the representation theory of these higher groupoids (relevant when extending field theories down to higher codimension)

What is different in higher gauge theory?

- Symmetries in “1-dimensional algebra” can be global or local
- Local symmetries are expressed by groupoids (e.g. of transport functors, in gauge theory)
- Global symmetries are expressed by group actions
- The two are related by transformation groupoids
- Transformation “groupoids” for 2-group actions are double categories (i.e. groupoids internal to **Cat**)
- Functor categories are naturally 2-groupoids
- Global and local symmetry are still related
- But there are local symmetries in higher gauge theory which are not global symmetries! (Not true in ordinary gauge theory)

# Global and Local Symmetry

Symmetry is a key concept in physical theories. It can be understood *globally* or *locally*[1]. (c.f. Weinstein)
Local symmetry relations of a set $X$[2] can be represented as a groupoid with:

- Objects: the elements of $X$
- Isomorphisms: $f : x \to y$ denoting a symmetry relation between $x$ and $y$

[1] Terminology conflict warning: this may clash with other standard uses of these words. "Global symmetries" in this sense turn out to consist of the ($n$-)group of *local* gauge transformations - later, we use *strict* and *costrict* for this notion

[2] For "set", we can, if careful, replace "object of a concrete category", such as **Top**, **Man**, etc.

Global symmetry involves group actions:

### Definition

A group action $\phi$ on a set $X$ is a functor $F : G \to \textbf{Sets}$ where the unique object of $G$ is sent to $X$. Equivalently, it is a function $\hat{F} : G \times X \to X$ which commutes with the multiplication (composition) of $G$:

$$\begin{CD} G \times G \times X @>{(1_G,\hat{F})}>> G \times X \\ @V{(m,1_X)}VV @VV{\hat{F}}V \\ G \times X @>>{\hat{F}}> X \end{CD} \tag{1}$$

Not all local symmetry situations come from a global one, but any group action gives a groupoid called the *transformation groupoid*.

### Definition

The **transformation groupoid** of an action of a group $G$ on a set $X$ is the groupoid $X /\!\!/ G$ with:

- **Objects**: All $x \in X$
- **Morphisms**: Pairs $(g, x) \in G \times X$, with $s(g, x) = x$, and $t(g, x) = F(g, x)$
- **Composition**: $(g', gx) \circ (g, x) = (g'g, x)$

Groupoids representing local symmetry need not be transformation groupoids.

Example

- If $M$ is a smooth manifold with an action of a Lie group $G$, take the full subgroupoid of $M /\!\!/ G$ on any open neighborhood $U \subset M$. Most are not transformation groupoids.
- Disjoint unions of transformation groupoids for different group actions (e.g. for a disconnected space with different symmetries on each connected component)

# Groupoids of Connections

One possible approach to higher gauge theory is by transport functors:

## Definition

A (flat) $G$-**connection** is a functor

$$A : \Pi_1(M) \to G$$

which assigns *holonomies* to paths in $M$. A **gauge transformation** $\alpha : A \to A'$ is a natural transformation (which assigns $\alpha_x \in G$ to each $x \in M$ with

$$\alpha_y A(\gamma) = A'(\gamma)\alpha_x$$

for each path $\gamma : x \to y$).

Flat connections and natural transformations form the objects and morphisms of the *groupoid of flat connections*

$$\mathcal{A}_0 M = Fun(\Pi_1(M), G)$$

Proposition

*If M is a connected manifold, $\mathcal{A}_0 M$ is equivalent to the transformation groupoid of an action of a group of all gauge transformations on the space of all connections:*

$$Conn /\!\!/ Gauge \cong Fun(\Pi_1(M), G) \tag{2}$$

This is the statement we want to generalize to 2-groups. For technical reasons, it is easier to give a discrete version of the result, but morally we have:

Theorem

*Given a manifold $(M)$, and a strict 2-group $\mathcal{G}$ presented by the crossed module $(G, H, \rhd, \partial)$, there is an isomorphism:*

$$\mathbf{Conn} /\!\!/ \mathbf{Gauge} \cong Hom_{\square}(\Pi_2(M), \mathcal{G}) \tag{3}$$

# 2-Groups and Crossed Modules

Goal: We want to construct an analog of $\mathbf{C}/\!\!/\mathcal{G}$ for an action of a 2-group.

### Definition

A **2-group** $\mathcal{G}$ is a 2-category with one object, and all morphisms and 2-morphisms invertible. A **categorical group** is a group object in **Cat**: a category $\mathcal{G}$ with $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and an inverse map satisfying the usual group axioms.

These are equivalent since a categorical group "is" a 2-group with one object.

2-groups are classified by crossed modules:

### Definition

A crossed module consists of $(G, H, \triangleright, \partial)$, where $G$ and $H$ are groups, $G \triangleright H$ is an action of $G$ on $H$ by automorphisms and $\partial : H \to G$ is a homomorphism, satisfying the equations:

$$\partial(g \triangleright \eta) = g\partial(\eta)g^{-1} \tag{4}$$

and

$$\partial(\eta) \triangleright \zeta = \eta\zeta\eta^{-1} \tag{5}$$

## Definition

The categorical group $\mathbf{G}$ given by $(G, H, \triangleright, \partial)$ has:

- **Objects**: $\mathbf{G}^{(0)} = G$
- **Morphisms**: $\mathbf{G}^{(1)} = G \times H$, with source and target maps

$$s(g, \eta) = g \text{ and } \qquad t(g, \eta) = \partial(\eta) g \tag{6}$$

- **Composition**:

$$(\partial(\eta) g, \zeta) \circ (g, \eta) = (g, \zeta\eta). \tag{7}$$

That is, as a group, $\mathbf{G}^{(1)} \cong G \ltimes H$, the semidirect product, with:

$$(g_1, \eta) \otimes (g_2, \zeta) = (g_1 g_2, \eta(g_1 \triangleright \zeta)) \tag{8}$$

# Higher Gauge Theory

Goal: Use 2-groups to generalize preceding constructions of connections and gauge transformations.

## Definition

If $M$ is a manifold with cell decomposition $\mathcal{D} = (V, E, F, \dots)$, then the *discrete fundamental 2-groupoid* $\Pi_2(M, \mathcal{D})$ is the 2-groupoid with

- *Objects*: the 0-cells of $V$
- *1-Morphisms*: $Hom(x, y)$ consists of 1-tracks in $M$ starting at $x \in V$ and ending at $y \in V$
- *2-Morphisms*: A 2-track $f : e \to e'$ between two 1-tracks is determined by a collection of faces $(f_1, f_2, \dots, f_l)$ in $f$, the composite of a sequence of homotopies between 1-tracks, each of the form

$$f_j' = (e_{i_1}, \dots, e_{i_n}) f_j (e_{i_{n+1}}, \dots, e_{i_m}) \tag{9}$$

(A discrete version of points, paths, and (thin) homotopy classes of

# 2-Groupoid of Connections

### Definition

The *gauge 2-groupoid* for a 2-group $\mathcal{G}$ on a manifold $M$ with cell decomposition $\mathcal{D}$ is:

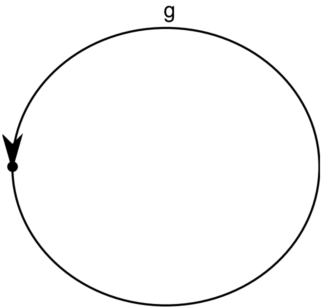
$$\mathcal{A}_0((M,\mathcal{D}),\mathcal{G}) = Hom_{\mathbf{Bicat}}(\Pi_2(M,\mathcal{D}),\mathcal{G}) \tag{10}$$

the 2-functor 2-category, which has:

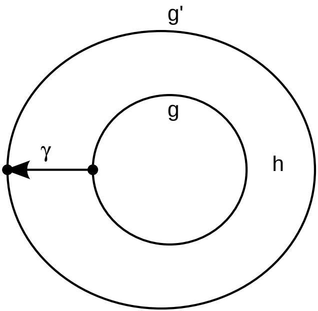
- Objects: 2-Functors from $\Pi_2(M)$ to $\mathcal{G}$, called **Connections**
- Morphisms: Pseudonatural transformations between functors, called **Gauge Transformations**
- 2-Morphisms: Modifications between pseudonatural transformations, called **Gauge Modifications**

(The term "gauge modification" appears not to be in common use yet!)

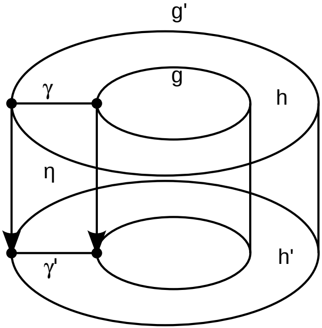
# Example 1: Connection on a Circle



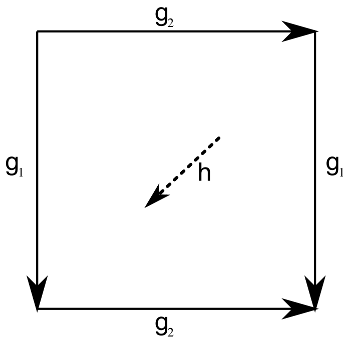
# Example 1: Gauge Transformation on a Circle



# Example 1: Gauge Modification on a Circle



# Example 2: Connection on a Torus



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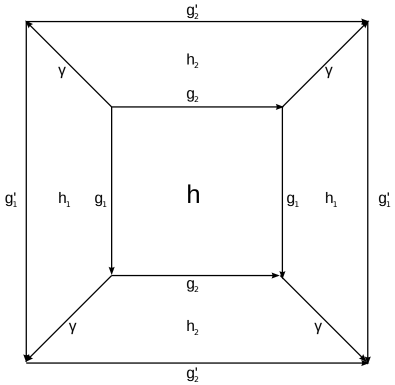

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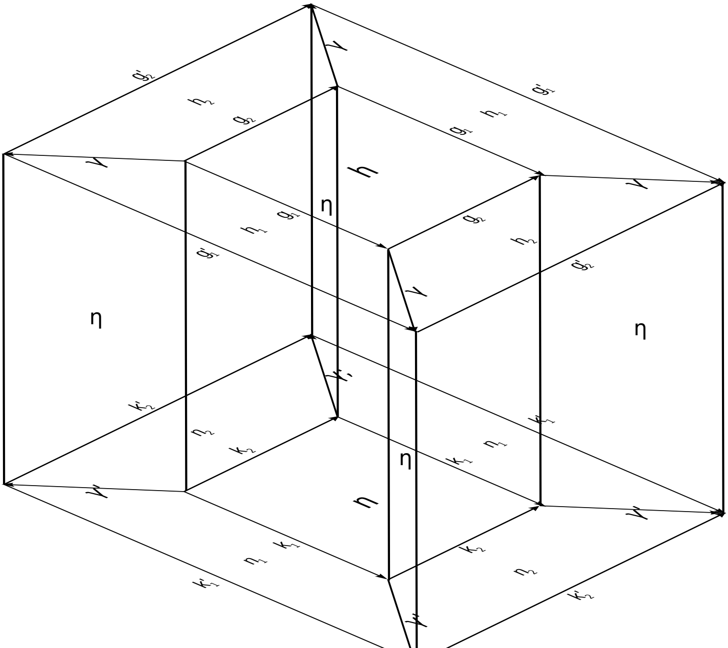


# Actions of 2-Groups on Categories

*Global* 2-group symmetry makes sense for objects in any bicategory:

## Definition

A 2-group $\mathcal{G}$ acts (strictly) on an object $\mathbf{C}$ in a bicategory $\mathcal{B}$ if there is a strict 2-functor:

$$\Phi : \mathcal{G} \to \mathcal{B}$$

whose image lies in $End(\mathbf{C})$.

In the case $\mathcal{B} = \mathbf{Cat}$:

- $\Phi(*) = \mathbf{C}$
- $\gamma \in Mor(\mathcal{G})$ gives an endofunctor:

$$\Phi_\gamma : \mathbf{C} \to \mathbf{C} \tag{11}$$

- $(\gamma, \eta) \in 2Mor(\mathcal{G})$ gives a natural transformation:

$$\Phi_{(\gamma,\eta)} : \Phi_\gamma \Rightarrow \Phi_{\partial(\eta)\gamma} \tag{12}$$

To make sense of *local* 2-group symmetries, we need an internal picture in **Cat**:

## Definition

A strict action of a categorical group $\mathcal{G}$ on a category $\mathbf{C}$ is a functor $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$ satisfying the action square diagram in **Cat** (strictly):

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} \times \mathbf{C} & \xrightarrow{\otimes \times Id_{\mathbf{C}}} & \mathcal{G} \times \mathbf{C} \\ {\scriptstyle Id_{\mathcal{G}} \times \hat{\Phi}} \downarrow & & \downarrow {\scriptstyle \hat{\Phi}} \\ \mathcal{G} \times \mathbf{C} & \xrightarrow[\hat{\Phi}]{} & \mathbf{C} \end{array} \tag{13}$$

## Lemma

*A strict 2-functor* $\Phi : \mathcal{G} \to End(\mathbf{C})$ *is equivalent to a strict action functor* $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$.

### Definition

If $\mathcal{G}$ is a categorical group classified by the crossed module $(G, H, \triangleright, \partial)$, and $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$ a strict action, let the notation $\blacktriangleright$ denote the following:

- Given $\gamma \in \mathcal{G}^{(0)} = G$ and $x \in \mathbf{C}^{(0)}$, let

$$\gamma \blacktriangleright x = \Phi_\gamma(x) = \hat{\Phi}(\gamma, x) \tag{14}$$

- Given $\gamma \in \mathcal{G}^{(0)} = G$ and $f \in \mathbf{C}^{(1)}$, let

$$\gamma \blacktriangleright f = \Phi_\gamma(f) = \hat{\Phi}((\gamma, 1_H), f) \tag{15}$$

- Given $(\gamma, \chi) \in \mathcal{G}^{(1)} = G \ltimes H$ and $(f : x \to y) \in \mathbf{C}^{(1)}$, let

$$\begin{aligned} (\gamma, \chi) \blacktriangleright f &= \hat{\Phi}((\gamma, \chi), f) \\ &= \Phi_{(\gamma,\chi)}(y) \circ (\gamma \blacktriangleright f) \\ &= (\partial(\chi)\gamma \blacktriangleright f) \circ \Phi_{(\gamma,\chi)}(x) \end{aligned} \tag{16}$$

# Transformation Double Categories

Idea: Since 2-group actions look just like actions of group objects, internal to **Cat**, we can construct the transformation groupoid in **Cat** as well.

A category **C** (in particular, a groupoid) internal in **Cat** has categories $C^{(0)}$ of objects and $C^{(1)}$ of morphisms. It is a *double category*, and we interpret the data of $C^{(0)}$ and $C^{(1)}$ as:

| | $C^{(0)}$ | $C^{(1)}$ |
|---|---|---|
| Objects | $x$ | $x \xrightarrow{f} y$ |
| Morphisms | $\begin{array}{c} x \\ \downarrow g \\ z \end{array}$ | $\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & \Swarrow F & \downarrow \\ z & \longrightarrow & w \end{array}$ |

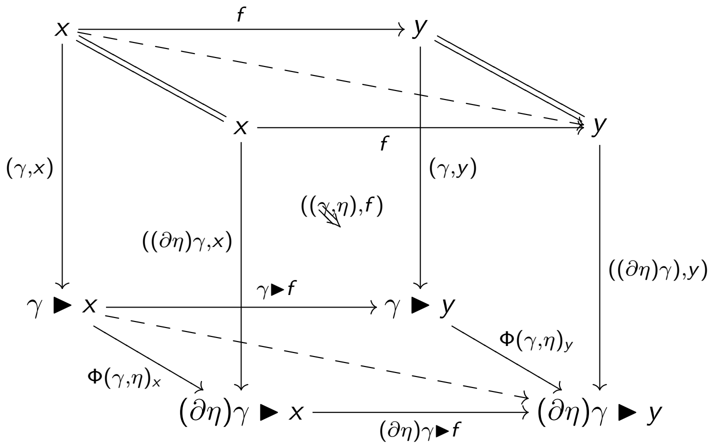
$\mathbf{C}/\!\!/\mathcal{G}$ is a category internal in **Cat**:
The category of objects is **C**, with objects and morphisms:

$$x \xrightarrow{f} y$$

The category of morphisms is $\mathbf{C} \times \mathcal{G}$, with source and target the projection and $\blacktriangleright$ respectively. Its objects and morphisms can be interpreted as the vertical arrows and squares of:

$$\begin{array}{ccc} x & \xrightarrow{\quad f \quad} & y \\ {\scriptstyle (\gamma,x)}\Big\downarrow & \Downarrow{\scriptstyle ((\gamma,\eta),f)} & \Big\downarrow{\scriptstyle ((\partial\eta)\gamma,y)} \\ \gamma \blacktriangleright x & \xrightarrow[(\gamma,\eta)\blacktriangleright f]{} & (\partial\eta)\gamma \blacktriangleright y \end{array}$$

The squares are diagonals of the naturality cubes:




(Note: the cube's faces are themselves special cases of squares when $f = Id_x$ or $\eta = 1_H$.)

# Structure of Transformation Groupoid

**Theorem**

*If $\mathcal{G}$ is given by a crossed module $(G, H, \partial, \triangleright)$, then $(\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(0)} = \mathbf{C}^{(0)}/\!\!/G$, and $\mathbf{C}^{(1)}/\!\!/G \subset (\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(1)} = \mathbf{C}^{(1)}/\!\!/(G \ltimes H)$.*

This lets us relate the three group actions represented by the overloaded symbol $\blacktriangleright$:

**Theorem**

*The identity-inclusion functor*

$$id : (\mathbf{C}/\!\!/\mathcal{G})^{(0)} \to (\mathbf{C}/\!\!/\mathcal{G})^{(1)} \tag{17}$$

*factors into two inclusions:*

$$(\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(0)} \subset \mathbf{C}^{(1)}/\!\!/\mathcal{G}^{(0)} \subset (\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(1)} \tag{18}$$

# Functor Double Category

Question: How is the transformation double category related to the functor 2-category depicted in our earlier pictures?

## Definition

A *pseudonatural transformation* $p : F \Rightarrow G$ between 2-functors assigns a **B**-morphism to each **A**-object, and a **B**-2-morphism to each **A**-morphism such that, for each **A**-morphism $f : x \to y$, the following square commutes up to the 2-cell filling it:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle p(x)}\downarrow & \swArrow{\scriptstyle p(f)} & \downarrow{\scriptstyle p(y)} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array} \tag{19}$$

Definition

A pseudonatural transformation $p : F \Rightarrow G$ is *strict* if this square commutes strictly, i.e.

$$p(f) = Id \tag{20}$$

and *costrict* if, $\forall x \in \mathbf{A}$

$$F(x) = G(x) \text{ and } \qquad p(x) \equiv Id_{F(x)} \tag{21}$$

The costrict transformations are denoted "ICONs" by Lack (an acronym for "Identity-Component Oplax Natural transformations").

### Definition

Given bi-groupoids **A** and **B**, define a double groupoid $Hom_{\square}(\mathbf{A}, \mathbf{B})$ with:

- **Objects**: (strict) Functors from **A** to **B**
- **Horizontal Morphisms**: Costrict transformations
- **Vertical Morphisms**: Strict transformations
- **Squares**: Modifications $M : s_2 \circ c_F \Rightarrow c_G \circ s_1$:

$$\begin{array}{ccc} F_1 & \xrightarrow{c_1} & G_1 \\ {\scriptstyle s_F}\downarrow & \Swarrow_{M} & \downarrow{\scriptstyle s_G} \\ F_2 & \xrightarrow[c_2]{} & G_2 \end{array} \tag{22}$$

### Theorem

*There is a 1-1 correspondence between modifications in $Hom(\mathbf{A}, \mathbf{B})$ and squares in $Hom_{\square}(\mathbf{A}, \mathbf{B})$.*

# Category of 2-Group Connections

### Definition (Category of Connections - Part 1)

The **category of connections**, $\mathbf{Conn} = \mathbf{Conn}(\mathcal{G}, (V, E, F))$, has the following:

- Objects of **Conn** consist of pairs of the form

$$\{(g, h) | g : E \to G,\ h : F \to H \text{ s.t. } \prod_{e \in \partial f} g(e) = \partial h(f)\}$$

- **Morphisms**: Morphisms of **Conn** with a given source $(g, h)$ are labelled by $\eta : E \to H$.

## Definition (Category of Connections - Part 2)

The target of a morphism from $(g, h)$ labelled by $\eta$ is $(g', h')$ with:

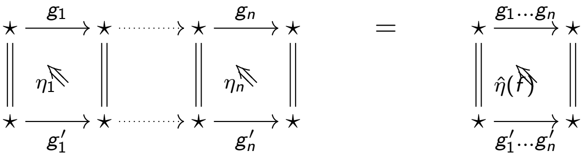
$$g'(e) = \partial(\eta(e))g(e)$$

and

$$h'(f) = h(f)\hat{\eta}(\partial(f))$$

The term $\hat{\eta}$ is the total $H$-holonomy around the boundary of the face $f$, whose edges are $e_i$ (taken in order):

$$\hat{\eta}(\partial(f)) = \prod_{e_j \in \partial(f)} (\prod_{i=1}^{j} g_i) \rhd \eta_j$$




(23)

# 2-Group of Gauge Transformations

Definition

Given $M$ with cell decomposition including $(V, E, F)$ as above, the **2-group of gauge transformations** is $\mathbf{Gauge} = \mathcal{G}^V$, which has:

- objects $\gamma : V \to G$
- morphisms $(\gamma, \chi)$ with $\chi : V \to H$
- 2-group structure given by $\partial$ and $\triangleright$ acting pointwise as in $\mathcal{G}$

Claim: there is a natural action of **Gauge** on **Conn**:

$$\Phi : \mathbf{Gauge} \to End(\mathbf{Conn})$$

# Action of **Gauge** on **Conn**

## Definition (Gauge 2-Group Action - Part 1)

The action of **Gauge** on **Conn** is given by:

- An object $\gamma : V \to G$ of **Gauge** gives a functor

$$\Phi(\gamma) : \mathbf{Conn} \to \mathbf{Conn}$$

  which acts via "conjugation by $\gamma$":

  - on objects $(g, h) \in$ **Conn** by:

$$\Phi(\gamma)(g, h) = (\hat{g}, \gamma \rhd h)$$

    where

$$\hat{g}(e) = \gamma(s(e))^{-1} g(e) \gamma(t(e))$$

    and

$$(\gamma \rhd h)(e) = \gamma(s(e_1)) \rhd h(f)$$

  - on morphisms $((g, h), \eta)$ by:

$$\Phi(\gamma)((g, h), \eta) = ((\hat{g}, \gamma \rhd h), \eta)$$

## Definition (Gauge 2-Group Action - Part 2)

- A morphism $(\gamma, \chi)$ of **Gauge** gives a natural transformation

$$\Phi(\gamma,\chi) : \Phi(\gamma) \Rightarrow \Phi(\gamma') : \textbf{Conn} \to \textbf{Conn}$$

  where $\gamma' = \partial(\chi)\gamma$, defined as follows: for each object $(g, h) \in$ **Conn**,

$$\Phi(\gamma,\chi)(g,h) = ((\tilde{g}, \tilde{h}), \tilde{\eta})$$

  where
  - $\tilde{g}(e) = \gamma(s(e))^{-1} g(e) \gamma(t(e))$
  - $\tilde{h}(f) = h(f)$
  - $\tilde{\eta}(e) = \gamma(s(e))^{-1} \rhd (\chi(s(e))^{-1}.g \rhd \chi(t(e)))$

  for each $e \in E$, $f \in F$.

Goal: 2-Group analog of the theorem that $\mathcal{A}_0 M$ is equivalent to a transformation groupoid, for this action.

# Main Theorem

**Theorem**

*Given a manifold with cell decomposition, $(M, \mathcal{D})$, with $\mathcal{D} = (V, E, F)$, and a strict 2-group $\mathcal{G}$ presented by the crossed module $(G, H, \triangleright, \partial)$, there is an isomorphism double functor $T$ of double groupoids:*

$$T : \mathbf{Conn} /\!\!/ \mathbf{Gauge} \cong Hom_{\Box}((\Pi_2(M), \mathcal{D}), \mathcal{G}) \tag{24}$$

(The definition of $T$ is given on the following slides.)

## Definition (Part I)

**Objects**: For $(g, h) \in \mathbf{Conn} /\!\!/ \mathbf{Gauge}$, define the 2-functor $T(g, h) : \Pi_2(M) \to \mathcal{G}$ by:

- On objects: $T(g, h)(v) = \star, \forall v \in V$
- On morphisms: a 1-track $e \in M^{(1)}$ is a thin equivalence class of edge paths in $\mathcal{D}$. If $e$ is represented by the sequence of edges $(e_1, e_2, \ldots, e_k)$, then let

$$T(g, h)(e) = g(e_1)g(e_2)\ldots g(e_k) \tag{25}$$

## Definition (Part II)

- On 2-morphisms: On each 2-track $f_1 \circ \cdots \circ f_n$ with

$$f_j' = (e_{i_1}, \ldots, e_{i_n}) f_j (e_{i_{n+1}}, \ldots, e_{i_m}) \tag{26}$$

define

$$T(g,h)(f_j') = (g(e_{i_1} \ldots g(e_{i_n})) h(f_j)(g(e_{i_{n+1}}), \ldots, g(e_{i_m})) \tag{27}$$

and by functoriality

$$T(g,h)(f) = T(g,h)(f_1) \ldots T(g,h)(f_l) \tag{28}$$

## Definition (Part III)

- **Horizontal Gauge Transformations**: A horizontal morphism in $((g,h)\xrightarrow{f_\eta}(g',h')) \in \mathbf{Conn}/\!\!/\mathbf{Gauge}$ is determined by $\eta : E \to H$, where $g' = \partial(\eta)g$. Then define the costrict transformation:

$$T(\eta) : T(g,h) \to T(g',h') \tag{29}$$

by

$$T(\eta)(e) = \eta(e_1)\eta(e_2)\ldots\eta(e_k) : T(g,h)(e) \Rightarrow T(g',h')(e) \tag{30}$$

## Definition (Part IV)

- **Vertical Gauge Transformations**: A vertical morphism in $((g, h), \gamma) \in \mathbf{Conn}/\!\!/\mathbf{Gauge}$ is a pair in $\mathbf{Conn} \times \mathbf{Gauge}$, (so $\gamma : V \to G$). Denote this by $\gamma$ for short, and define the strict natural transformation:

$$T(\gamma) : T(g, h) \to T(\hat{g}, \gamma \triangleright h) \tag{31}$$

by $T(\gamma)(v) = \gamma(v)$.

## Definition (Part V)

- **Gauge Modifications**: Recall that a gauge modification is a square in **Conn**$/\!\!/$**Gauge**, a morphism in the morphism category, which is determined by a pair of morphisms $(((g,h),\eta),(\gamma,\chi)) \in$ **Conn** $\times$ **Gauge**. Denote this $\chi$ for short, and define the modification

$$T(\chi) : T(\gamma)T(\partial(\chi)\eta) \Rightarrow T(\eta)T(\hat{\Phi}(\eta,\chi)) \qquad (32)$$

or equivalently

$$T(\chi) : T(\gamma)T(\partial(\chi)\eta) \Rightarrow T(\eta)T(\Phi_{\chi}(g,h)\Phi_{(\partial\chi)\gamma}(\eta)) \qquad (33)$$

It is just defined by $T(\chi)(v) = \chi(v)$.

# Example 1: Connections on the Circle

$\mathcal{A}_0 S^1 = Hom(\Pi_2(S^1), \mathcal{G})$, with:

- Objects: Functors $F : \Pi_2(S^1) \to \mathcal{G}$, which are determined by $F(1) \in G$
- Morphisms: Natural transformations $n : F \Rightarrow F'$ determined by $\gamma \in G$ and $\eta \in H$
- 2-Morphisms: Modifications $\phi : n \Rightarrow n'$ determined by $\chi \in H$

## Theorem

*There is an equivalence of 2-groupoids $\mathcal{A}_0 S^1 \cong \mathcal{G} /\!\!/ \mathcal{G}$.*

# Generalization to $n$-Groups

This phenomenon should generalize to $n$-groups for $n > 2$. Some straightforward conjectures:

- $(k+1)$-group gauge theory should give moduli space as a $k$-groupoid internal to **kCat** as "transformation groupoid"
- For $n = 3$ ($k = 2$) this is a "double bicategory"
- Should relate to global symmetries by a $n$-group action on a $n$-category
- In general, a square array of morphism types
- All these have *local* descriptions in terms of forms on spacetime: graded by form degree and morphism degree of the gauge $k$-group
- Should be a bicomplex with compatible crossed-complex structures in each direction (since crossed complex $\cong$ $k$-groupoids)