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Dirac Sigma Models

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Abstract: We introduce a new topological sigma model, whose fields are bundle maps from the tangent bundle of a 2-dimensional world-sheet to a Dirac subbundle of an exact Courant algebroid over a target manifold. It generalizes simultaneously the (twisted) Poisson sigma model as well as the G/G-WZW model. The equations of motion are satisfied, iff the corresponding classical field is a Lie algebroid morphism. The Dirac Sigma Model has an inherently topological part as well as a kinetic term which uses a metric on worldsheet and target. The latter contribution serves as a kind of regulator for the theory, while at least classically the gauge invariant content turns out to be independent of any additional structure. In the (twisted) Poisson case one may drop the kinetic term altogether, obtaining the WZ-Poisson sigma model; in general, however, it is compulsory for establishing the morphism property.

Contents

1. Introduction

In this paper we introduce a new kind of two-dimensional topological sigma model which generalizes simultaneously the Poisson Sigma Model (PSM) [27, 15, 16] and the G/G WZW model [11, 12] and which corresponds to general Dirac structures [8, 22] (in exact Courant algebroids). Dirac structures include Poisson and presymplectic structures as particular cases. They are dim M-dimensional subbundles D of $E := T^*M \oplus TM$ satisfying some particular properties recalled in the body of the paper below. If one regards the graph of a contravariant 2-tensor $\mathcal{P} \in \Gamma(TM^{\otimes 2})$ viewed as a map from T^*M to TM, then $D = \text{graph}\,\mathcal{P}$ turns out to be a Dirac structure if and only if $\mathcal{P}$ is a Poisson bivector (i.e. $\{f, g\} := \mathcal{P}(\mathrm{d}f, \mathrm{d}g)$, $f, g \in C^\infty(M)$, defines a Poisson bracket on M). Likewise, $D := \text{graph}\,\omega$, where ω is a covariant 2-tensor viewed as a map $TM \to T^*M$ is a Dirac structure, iff ω is a closed 2-form.

More generally, the above construction is twisted by a closed 3-form H, and in addition not any Dirac structure can be written as a graph from T^*M to TM or vice versa. This is already true for a Dirac structure that can be defined canonically on any semi-simple Lie group G and which turns out to govern the G/G-WZW model. Only after cutting out some regions in the target G of this σ-model, the Dirac structure D is the graph of a bivector and the G/G model can be cast into the form of a (twisted) PSM [3, 10]. The new topological sigma model we are suggesting, the Dirac Sigma Model (DSM), works for an arbitrary Dirac structure.

We remark parenthetically that also generalized complex structures, which lately have received increased attention in string theory, fit into the framework of Dirac structures. In this case one regards the complexification of E and, as an additional condition, the Dirac structure D called a "generalized complex structure" needs to have trivial intersection with its complex conjugate. The focus of this text is on real E, but we intend to present an adaptation separately (for related work cf. also [35, 21, 4]).

As is well-known, the quantization of the PSM yields the quantization of Poisson manifolds [18, 6] (cf. also [26]). In particular, the perturbative treatment yields the Kontsevich formula. The reduced phase space of the PSM on a strip carries the structure of a symplectic groupoid integrating the chosen Poisson Lie algebroid [7]. One may expect to obtain similar relations for the more general DSM. Also, several two-dimensional field theories of physical interest were cast into the form of particular PSMs [27, 17, 30, 14] and thus new efficient tools for their analysis were accessible. The more general DSMs should permit to enlarge this class of physics models.

The definition of the DSM requires some auxiliary structures. In particular one needs a metric g and h on the target manifold M and on the base or worldsheet manifold Σ, respectively. The action of the DSM consists of two parts, $S_{\mathrm{DSM}} = S_{\mathrm{top}} + S_{\mathrm{kin}}$, where only the "kinetic" term S_{kin} depends on g and h. If $D = \text{graph}\,\mathcal{P}$, S_{kin} may be dropped, at least classically, in which case one recovers the PSM (or its relative, twisted by a closed 3-form, the WZ-Poisson Sigma Model). We conjecture that for what concerns the *gauge invariant* information captured in the model on the classical level one may always drop S_{kin} in S_{DSM}—and for $\Sigma \cong S^1 \times \mathbb{R}$ we proved this, cf. Theorem 4 below. Still, even classically, it plays an important role, serving as a kind of regulator for the otherwise less well behaved topological theory. E.g., in general, it is only the presence of S_{kin} which ensures that the field equations of S_{DSM} receive the mathematically appealing interpretation of Lie algebroid morphisms from $T\Sigma$ to the chosen Dirac structure D—in generalization of an observation for the PSM [5]. (We will recall these notions in the body of the paper, but mention already here that $T\Sigma$ as well as any Dirac structure canonically carry a Lie algebroid structure). Without S_{kin}, the solutions of the Euler Lagrange equations constrain the fields less in general, which then seems to be balanced by additional gauge symmetries broken by S_{kin}. These additional symmetries can be difficult to handle mathematically, since in part they are supported on lower dimensional regions in the target of the σ-model.

The paper is organized as follows: In Sect. 2 we use the G/G model as a starting point for deriving the new sigma model. This is done by rewriting the G/G-WZW model in terms suitable for a generalization. By construction, the generalization will be such that the PSM is included, up to S_{kin}, as mentioned above. The role of the Poisson bivector $\mathcal{P}$ in the PSM is now taken by an orthogonal operator $\mathcal{O}$ on TM, which in the Poisson case is related to $\mathcal{P}$ by a Cayley transform, but which works in the general case.

In Sect. 3 we provide the mathematical background that is necessary for a correct interpretation of the structures defining the general sigma model. This turns out to be the realm of Courant algebroids and Dirac structures. We recapitulate definitions and facts known in the mathematics literature, but also original results, developed to address the needs of the sigma model, are contained in this section. The action of the Dirac sigma model is then recognized as a particular functional on the space of vector bundle morphisms $\phi\colon T\Sigma \to D$, $S_{\text{DSM}} = S_{\text{DSM}}[\phi]$. Specializing this to the PSM, one reproduces the usual fields $\tilde{\phi}\colon T\Sigma \to T^*M$, since precisely in this case D is isomorphic to T^*M.

In Sect. 4 we point out that the definition of the DSM presented in the preceding sections also depends implicitly on some further auxiliary structure in addition to the chosen Dirac structure $D \subset E$, namely on a "splitting" in the exact Courant algebroid $0 \to T^*M \to E \to TM \to 0$. This dependence occurs in S_{top}, but again at the end of the day, the "physics" will not be affected by it.

In Sect. 5 we derive the field equations of S_{DSM}, which we present in an inherently covariant way. We also prove that ϕ solves the field equation, iff it respects the canonical Lie algebroid structures of $T\Sigma$ and the Dirac structure D, i.e. iff $\phi\colon T\Sigma \to D$ is a Lie algebroid morphism. We present one possible covariant (global and frame independent) form of the gauge symmetries of S_{DSM}, furthermore, using the connection on M induced by the auxiliary metric g. We will, however, postpone the corresponding proof of the gauge invariance and further interpretations to another work [20], where the question of covariant gauge symmetries will be addressed in a more general framework of Lie algebroid theories, for which the DSM serves as one possible example. There we will also relate these symmetries to the more standard presentations of the symmetries of the G/G model and the (WZ-)PSM.

In Sect. 6, finally, which in most parts can be read also directly after Sect. 2, we determine the Hamiltonian structure of the DSM. In fact, we will do so even for a somewhat more general sigma model, where the target subbundle $D \subset E$ is not necessarily required to be integrable. It turns out that the constraints of this model are of the form introduced recently in [2], where now currents J are associated to any section $\psi \in \Gamma(D)$. As a consequence of the general considerations in [2], the constraints $J_\psi = 0$ are found to be first class, iff D is integrable, i.e. iff it is a Dirac structure.

2. From G/G to Dirac Sigma Models

We will use the G/G-WZW-model [11, 12] as a guide to the new sigma model that is attached to any Dirac structure. Given a Lie group G with quadratic Lie algebra $\mathfrak{g}$ and a closed 2-manifold Σ equipped with some metric h, which we assume to be of Lorentzian signature for simplicity, the (multivalued) action functional of the WZW-model consists of two parts (cf. [33] for further details): a kinetic term for G-valued fields $\mathrm{g}(x)$ on $\Sigma \ni x$ as well as a Wess-Zumino term S_{WZ}, requiring the (non-unique) extension of $im\Sigma \subset \mathrm{G}$ to a 3-manifold $N_3 \subset \mathrm{G}$ such that $\partial N_3 = im\Sigma$:

$$S_{\text{WZW}}[\mathrm{g}] = \frac{k}{4\pi} \int_\Sigma \langle \partial_+ \mathrm{g}\mathrm{g}^{-1}, \partial_- \mathrm{g}\mathrm{g}^{-1} \rangle \, \mathrm{d}x^- \wedge \mathrm{d}x^+ + S_{WZ}, \qquad (1)$$

$$S_{\mathrm{WZ}}[\mathrm{g}] = \frac{k}{12\pi} \int_{N_3} \langle \mathrm{dgg}^{-1} \stackrel{\wedge}{,} (\mathrm{dgg}^{-1})^{\wedge 2} \rangle \,, \tag{2}$$

where x^+, x^- are lightcone coordinates on Σ (i.e. $h = \rho(x^+, x^-)\, \mathrm{d}x^+ \mathrm{d}x^-$ for some locally defined positive function ρ), $\langle \cdot, \cdot \rangle$ denotes the Ad-invariant scalar product on $\mathfrak{g}$, and k is an integer multiple of $\hbar$ (which implies that the exponent of $\frac{i}{\hbar} S_{\mathrm{WZW}}$, the integrand in a path integral, is a unique functional of $\mathrm{g} \colon \Sigma \to \mathrm{G}$). Introducing a connection 1-form a on Σ with values in a Lie subalgebra $\mathfrak{h} < \mathfrak{g}$, one can lift the obvious rigid gauge invariance of (1) w.r.t. $\mathrm{g} \mapsto \mathrm{Ad}_{\mathrm{h}} \mathrm{g} \equiv \mathrm{hgh}^{-1}$, $\mathrm{h} \in \mathrm{H} < \mathrm{G}$, to a local one ($\mathrm{h} = \mathrm{h}(x)$ arbitrary, $a \mapsto \mathrm{hdh}^{-1} + \mathrm{Ad}_{\mathrm{h}} a$) by adding to S_{WZW},

$$S_{\mathrm{gauge}}[\mathrm{g}, a] = \frac{k}{2\pi} \int_\Sigma \Big(\langle a_+, \partial_- \mathrm{gg}^{-1} \rangle - \langle a_-, \mathrm{g}^{-1} \partial_+ \mathrm{g} \rangle + \langle a_+, a_- \rangle$$
$$- \langle a_+, \mathrm{g} a_- \mathrm{g}^{-1} \rangle \Big)\, \mathrm{d}^2 x \,, \tag{3}$$

where $\mathrm{d}^2 x \equiv \mathrm{d}x^- \wedge \mathrm{d}x^+$. For the maximal choice $\mathrm{H} = \mathrm{G}$ this yields the G/G-model:

$$S_{\mathrm{G/G}}[\mathrm{g}, a] = S_{\mathrm{WZW}}[\mathrm{g}] + S_{\mathrm{gauge}}[\mathrm{g}, a] \,. \tag{4}$$

In [3] it was shown that on the Gauss decomposable part $\mathrm{G}_{\mathrm{Gauss}}$ of G (for $SU(2)$ this is all of the 3-sphere except for a 2-sphere) the action (4) can be replaced equivalently by a Poisson Sigma Model (PSM) with target $\mathrm{G}_{\mathrm{Gauss}}$. (This was re-derived in a more covariant form in [10]). It is easy to see that by similar manipulations—and in what follows we will demonstrate this by a slightly different procedure—(4) can be cast into a WZ-PSM [16] on $\mathrm{G}_1 := \mathrm{G} \backslash \mathrm{G}_0$, where $\mathrm{G}_0 = \{\mathrm{g} \in \mathrm{G} \,|\, \ker(1 + \mathrm{Ad}_{\mathrm{g}}) \neq \{0\}\}$ (again a 2-sphere for $SU(2)$). The question may arise, if there do not exist possibly some other manipulations that can cast the G/G-model into the form of a WZ-PSM *globally*, with a 3-form H of the same cohomology as the Cartan 3-form (the integrand of (2)). In fact this is not possible: it may be shown [1] that there is a cohomological obstruction for writing the Dirac structure which governs the G/G-model and which is disclosed below (the Cartan-Dirac structure, cf. Example 3 below) globally as a graph of a bivector. Consequently, this calls for a new type of topological sigma model that can be associated to *any* Dirac structure D (in an exact Courant algebroid) such that it specializes to the WZ-PSM if D may be represented as the graph of a bivector and e.g. to the G/G model if the target M is chosen to be G and D the Cartan-Dirac structure.[1]

Vice versa, the G/G model already provides a possible realization of the sought-after sigma model for this particular choice of M and D. We will thus use it to derive the new sigma model within this section. For this purpose it turns out to be profitable to rewrite (4) in the language of differential forms, so that the dependence on the world-sheet metric h becomes more transparent; for simplicity we put $k = 4\pi$ in what follows (corresponding to a particular choice of $\hbar$). We then find

$$S_{\mathrm{G/G}} = \frac{1}{2} \int_\Sigma \langle \mathrm{dgg}^{-1} \stackrel{\wedge}{,} *\mathrm{dgg}^{-1} \rangle + S_{\mathrm{WZ}} + \int_\Sigma \langle a \stackrel{\wedge}{,} *a \rangle - \langle a \stackrel{\wedge}{,} \mathrm{Ad}_{\mathrm{g}}(* - 1) a \rangle \tag{5}$$

$$+ \int_\Sigma \langle a \stackrel{\wedge}{,} (* - 1) \mathrm{dgg}^{-1} \rangle - \langle \mathrm{Ad}_{\mathrm{g}} a \stackrel{\wedge}{,} (* + 1) \mathrm{dgg}^{-1} \rangle \,. \tag{6}$$

[1] We remark parenthetically that in an Appendix in [3] it was shown that the G/G model can be represented on all of G as what we would call these days a WZ-PSM, but this was at the expense of permitting a *distributional* 3-form H (the support of which was on $\mathrm{G} \backslash \mathrm{G}_{\mathrm{Gauss}}$). The above mentioned topological obstruction relies on the smooth category.

Here our conventions for the Hodge dual operator $*$, which is the only place where h enters, is such that $*\mathrm{d}x^{\pm} = \pm\mathrm{d}x^{\pm}$. Now we split $S_{\mathrm{G/G}}$ into terms containing $*$ and those which do not. One finds that the first type of terms combines into a total square and that

$$S_{\mathrm{G/G}}[\mathrm{g}, a] = S_{\mathrm{kin}} + S_{\mathrm{top}}, \tag{7}$$

$$S_{\mathrm{kin}} = \frac{1}{2}\int_{\Sigma} ||\mathrm{dgg}^{-1} + (1 - \mathrm{Ad}_{\mathrm{g}})a||^2, \tag{8}$$

$$S_{\mathrm{top}} = \int_{\Sigma} \langle -(1 + \mathrm{Ad}_{\mathrm{g}})a \overset{\wedge}{,} \mathrm{dgg}^{-1}\rangle + \langle a \overset{\wedge}{,} \mathrm{Ad}_{\mathrm{g}}a\rangle + S_{\mathrm{WZ}}, \tag{9}$$

where for a Lie algebra valued 1-form β we use the notation $||\beta||^2 \equiv \langle \beta \overset{\wedge}{,} *\beta\rangle$.

Before generalizing this form of the action, we show that G/G can be cast into the form of a WZ-PSM (or HPSM for a given choice of H) on G_1. For this purpose we briefly recall the action functional of the WZ-PSM [16] (cf. also [24]): Given a closed 3-form H and a bivector field $\mathcal{P} = \frac{1}{2}\mathcal{P}^{ij}(X)\,\partial_i \wedge \partial_j$ on a target manifold M one considers

$$S_{\mathrm{HPSM}}[X, A] = \int_{\Sigma} A_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\mathcal{P}^{ij} A_i \wedge A_j + \int_{N_3} H\,, \tag{10}$$

where $X\colon \Sigma \to M$ and $A \in \Gamma(T^*\Sigma \otimes X^*T^*M)$ and the last term is again a WZ-term (i.e. $N_3 \subset M$ is chosen such that its boundary agrees with the image of X and the usual remarks about multi-valuedness of the functional can be made)[2]; for the case that $H = \mathrm{d}B$ the latter contribution can be replaced by (the single-valued) $\int_{\Sigma} \frac{1}{2} B_{ij}\, \mathrm{d}X^i \wedge \mathrm{d}X^j$. This kind of theory is topological (has a finite dimensional moduli space of classical solutions modulo gauge transformations) iff the couple $(\mathcal{P}, H)$ satisfies a generalization of the Jacobi-identity, namely

$$\mathcal{P}^{il}\partial_l\mathcal{P}^{jk} + \mathrm{cycl}(ijk) = H_{i'j'k'}\,\mathcal{P}^{i'i}\mathcal{P}^{j'j}\mathcal{P}^{k'k}\,; \tag{11}$$

$(\mathcal{P}, H)$ then defines a WZ-Poisson structure on M (also called "twisted Poisson" or "H-Poisson" or "Poisson with background" in the literature).

We now want to show that when restricting g to maps $\mathrm{g}\colon \Sigma \to \mathrm{G}_1$, the action $S_{\mathrm{G/G}}$ can be replaced by (10) for a particular choice of $\mathcal{P}$ and H, at least for what concerns the classical field equations. Let us consider the variation of the two contributions to $S_{\mathrm{G/G}}$ in (4) with respect to a separately:

$$-\mathrm{Ad}_{\mathrm{g}}\frac{\delta S_{\mathrm{kin}}}{\delta a} = (1 - \mathrm{Ad}_{\mathrm{g}}) * \left[\mathrm{dgg}^{-1} + (1 - \mathrm{Ad}_{\mathrm{g}})a\right], \tag{12}$$

$$-\mathrm{Ad}_{\mathrm{g}}\frac{\delta S_{\mathrm{top}}}{\delta a} = (1 + \mathrm{Ad}_{\mathrm{g}}) \left[\mathrm{dgg}^{-1} + (1 - \mathrm{Ad}_{\mathrm{g}})a\right]. \tag{13}$$

[2] In particular, assume that Σ has no boundary and is orientable. Assume furthermore that $H_2(M)$ is trivial and that $[H] \in H^3(M, 2\pi\hbar\mathbb{Z})$. Then one can give meaning to the *exponential* of $i S_{\mathrm{HPSM}}[X, A]/\hbar$, which is the integrand of a "path integral": Choose *any* (possibly degenerate) 3-manifold $N_3 \subset M$ with boundary $\mathrm{im}(X)$ to perform the integral. Note that the result $i S_{\mathrm{HPSM}}[X, A]/\hbar$ does not only depend on the cohomology class of H, but also the representative. For the field equations of such a "functional" S_{HPSM}, H need not have integral cycles, moreover, and we may drop the conditions $[H] \in H^3(M, 2\pi\hbar\mathbb{Z})$ and $H_2(M) = 0$, since only infinitesimal variations are needed. Likewise, a Hamiltonian formulation exists for any closed H, cf. Sect. 6. If Σ has a boundary, additional data need to be specified on "D-branes" in M; for the corresponding Hamiltonian formulation, cf. e.g. [2].

Decomposing the term in the square bracket into its ± 1 eigenvalues of $*$ (use projectors $\frac{1}{2}(1 \pm *)$ or, equivalently, consider the $\mathrm{d}x^+$ and $\mathrm{d}x^-$ components of (13)), it is easy to see that $\delta S_{\mathrm{G/G}}/\delta a = 0$ yields

$$\mathrm{dgg}^{-1} + (1 - \mathrm{Ad_g})a = 0\,. \tag{14}$$

On the other hand, as obvious from (13), this equation is also obtained from S_{top} on G_1. Since S_{kin} is quadratic in the left-hand side of (14), it gives no contribution to the variation of $S_{\mathrm{G/G}}$ w.r.t. g, which proves the desired equivalence.

For completeness we remark that the second field equation is nothing but the zero curvature condition[3]

$$F \equiv \mathrm{d}a + a \wedge a = 0\,, \tag{15}$$

and that this equation results from variation of S_{top} w.r.t. g even on all of G. Note however that S_{top} will have solutions mapping into $\mathrm{G}_0 = \mathrm{G}\backslash\mathrm{G}_1$ violating (14).

It thus remains to cast S_{top} into the form (10). On G_1 this is done most easily by introducing $A := -(1 + \mathrm{Ad_g})a$, where the matrix components of this 1-form correspond to a right-invariant basis of $T^*\mathrm{G}$ (note also that sections of $T^*\mathrm{G}$ and $T\mathrm{G}$, 1-forms and vector fields on the group, can be identified by means of the Killing metric); then the first term in (9) already takes the form of the first term in (10). The WZ-terms can be identified without any manipulations. It remains to calculate the bivector upon comparison of the respective second terms. A very simple calculation then yields

$$\mathcal{P} = \frac{1 - \mathrm{Ad_g}}{1 + \mathrm{Ad_g}}\,, \quad H = \frac{1}{3}\langle \mathrm{dgg}^{-1} \overset{\wedge}{,} (\mathrm{dgg}^{-1})^{\wedge 2}\rangle\,, \tag{16}$$

where $\mathcal{P}$ refers to a right-invariant basis on G again and H is the Cartan 3-form. In [29] it was shown that the WZ-Poisson structures [16] are particular Dirac structures; and the utility of this reformulation was stressed due to the simplification of checking (11) for the above example. We want to use the opportunity to stress the usefulness of sigma models in this context (cf. also [31]): Using the well-known fact that the G/G model is topological (in the sense defined above) and that one can cast it into the form (10) is already sufficient to establish (11) for (16); even more, the above consideration is a possible route for finding this example of a WZ-Poisson or Dirac structure. The above bivector also plays a role in the context of D-branes in the WZW-model.

We now come to the generalization of the G/G model written in the form (7). For this purpose we first rewrite (10) into a form more suitable to the language of Dirac structures. It is described as the graph of the bivector $\mathcal{P}$ in the bundle $E = T^*M \oplus TM$, i.e. as pairs $(\alpha, \mathcal{P}(\alpha, \cdot))$ for any $\alpha \in T^*M$. With the 1-forms A taking values in T^*M we thus may introduce the dependent 1-form $V = \mathcal{P}(A, \cdot)$ taking values in TM. Together they may be viewed as a 1-form $\mathcal{A} = A \oplus V$ on Σ taking values in the subbundle $D = \mathrm{graph}(\mathcal{P}) \subset E$. Then (10) can be rewritten as

$$S_{\mathrm{top}}[X, \mathcal{A}] = \int_\Sigma A_i \wedge \mathrm{d}X^i - \tfrac{1}{2} A_i \wedge V^i + \int_{N_3} H. \tag{17}$$

Comparison with (9) shows that in the G/G-model $A_\alpha = -\left[(1 + \mathrm{Ad_g})a\right]_\alpha$, where α is an index referring to a right-invariant basis on G. Correspondingly, we can read off by comparison of (9) with (17) that then $V^\alpha = -\left[(1 - \mathrm{Ad_g})a\right]^\alpha$ (showing equality is

[3] Written in a matrix representation. More generally, $F \equiv \mathrm{d}a + \frac{1}{2}[a \overset{\wedge}{,} a]$.

a simple exercise where one uses that Ad_g is an orthogonal operator w.r.t. the Killing metric). Note that in the formulation (17) no metric on M appears anymore; the Killing metric is used only in the above identification.

The G/G-model also contains a second part, which uses a metric h on Σ as well as a metric g on M. From the above identification it is easy to generalize it:

$$S_{\mathrm{kin}}[X, \mathcal{A}] = \frac{\alpha}{2} \int_\Sigma ||\mathrm{d}X - V||^2 \,, \tag{18}$$

where for any TM-valued 1-form $f = f^i \partial_i = f^i_\mu \mathrm{d}x^\mu \otimes \partial_i$ on Σ we use

$$||f||^2 := \mathrm{g}(f \stackrel{\wedge}{,} *f) \equiv \mathrm{g}_{ij} h^{\mu\nu} f^i_\mu f^j_\nu \, \mathrm{vol}_\Sigma \,, \quad \mathrm{vol}_\Sigma \equiv \sqrt{\det(h_{\mu\nu})}\mathrm{d}^2 x \,, \tag{19}$$

and where α is some coupling constant. For the action functional of the Dirac Sigma Model (DSM) we thus postulate $S_{\mathrm{DSM}}[X, \mathcal{A}] := S_{\mathrm{kin}} + S_{\mathrm{top}}$, i.e.

$$S_{\mathrm{DSM}}[X, A \oplus V] = \frac{\alpha}{2} \int_\Sigma ||\mathrm{d}X - V||^2 + \int_\Sigma A_i \wedge \mathrm{d}X^i - \tfrac{1}{2} A_i \wedge V^i + \int_{N_3} H \,. \tag{20}$$

As already mentioned above, the 1-forms $\mathcal{A} \equiv A \oplus V$ take value in X^*D, where D is a Dirac structure; this will be made more precise and explicit below. For Lorentzian signature metrics h on Σ, α should be real and (preferably) non-vanishing; for Euclidean signatures of h we need an imaginary unit as a relative factor between the kinetic and the topological term. Although possibly unconventional, we will include it in the coupling constant α in front of the kinetic term, so that we are able to cover all possible signatures in one and the same action functional. If g has an indefinite signature, on the other hand, we in addition need to restrict to a neighborhood of the original value $\alpha = 1$ (or $\alpha = i$ for Euclidean h); the condition we want to be fulfilled is the invertibility of the operator (42) below (cf. also the text following Corollary 2).

The metrics h and g on Σ and M, respectively, are of auxiliary nature. First of all, it is easy to see that S_{kin} gives *no* contribution to the field equations for what concerns the WZ-PSM (10); also the gauge symmetries are modified only slightly by some on-shell vanishing, and thus physically irrelevant contribution (both statements will be proven explicitly in subsequent sections). Let us consider the other extreme case of a Dirac structure provided by the subbundle $D = TM$ to $E = T^*M \oplus TM$ (for H being zero): In this case $A \equiv 0$ and $V = V^i \partial_i$ is an unconstrained 1-form field. Obviously in this case $S_{\mathrm{top}} \equiv 0$ and one obtains *no* field equations from this action alone. On the other hand, the field equations from S_{kin} are computed easily as

$$\mathrm{d}X^i = V^i \,. \tag{21}$$

First we note that this equation does not depend either on h or on g; these two structures are of auxiliary nature for obtaining a nontrivial field equation in this case, a fact that will be proven also for the general case in the subsequent section (cf. Theorem 1 below). Secondly, we observe that at the end of the day even in this case the two theories $S_{\mathrm{top}}[X, \mathcal{A}] \equiv 0$ and $S_{\mathrm{DSM}}[X, \mathcal{A}] \equiv S_{\mathrm{kin}}[X, V]$ are still not so different as one may expect at first sight: The *moduli space* of classical solutions is the same for both theories. The lack of field equations in the first case is compensated precisely in the correct way by additional gauge symmetries, that are absent for $S_{\mathrm{kin}}[X, V]$. If we permit as gauge symmetries those that are in the connected component of unity, we find the homotopy classes $[X]$ of $X\colon \Sigma \to M$ as the only physically relevant information.

(Gauge identification of different homotopy classes might be considered as large gauge transformations, as both action functionals remain unchanged in value; then the moduli space would be just a point in each case).

One may speculate that this mechanism of equivalent moduli spaces occurs also in the general situation. We leave this as a conjecture for a general choice of Σ, proving it in the case of $\Sigma = S^1 \times \mathbb{R}$, where we will establish equivalent Hamiltonian structures (cf. Sect. 6 below). We remark, however, that the equivalence may require some slightly generalized notion of gauge symmetries similar to the λ-symmetry discussed in [32]; this comes transparent already from the G/G example, where the additional classical solutions found above, which are located at regions in G where the kernel of $1 + \mathrm{Ad}_{\mathrm{g}}$ is nontrivial, need to be gauge identified by *additional* gauge symmetries of S_{top} that are concentrated at the same region in G.

Note that this complication disappears when S_{kin} is added to S_{top}. Likewise, the field equations (21) have a nice mathematical interpretation; they are equivalent to the statement that the fields $(X, \mathcal{A})$ are in one-to-one correspondence with morphisms from $T\Sigma$ to D, both regarded as Lie algebroids (cf. Theorem 1 below). So, the addition of S_{kin} (with non-vanishing α) serves as a kind of regulator for the theory, making it mathematically more transparent and more tractable–while simultaneously the "physics" (moduli space of solutions) seems to remain unchanged in both cases, $\alpha = 0$ and $\alpha \neq 0$.

Having an auxiliary metric g on M at one's disposal, it may profitably be used to reformulate S_{top}. In particular, it will turn out that one may use it to parameterize the Dirac structure *globally* in terms of an orthogonal operator $\mathcal{O}\colon TM \to TM$ (cf. Proposition 1 below), generalizing the operator Ad_{g} on $M = \mathrm{G}$ in the G/G model above; this then permits one to use unrestricted fields for the action functional again, such as g and a in the G/G model and X and A in the WZ-PSM.

Essentially, this works as follows: By means of g we may identify T^*M with TM, so $E \cong TM \oplus TM$ and the parts A and V of $\mathcal{A} = A \oplus V$ may be viewed both as (1-form valued) vector (or covector) fields on M (corresponding to the index position, where indices are raised and lowered by means of g). Introducing the involution $\tau\colon E \to E$ that exchanges both copies of TM, $\tau(\alpha \oplus v) = v \oplus \alpha$, let us consider its eigenvalue subbundles $E_\pm = \{(v \oplus \pm v)\,,\ v \in TM\}$, both of which can be identified with TM by projection to the first factor $T^*M \cong TM$. It turns out (cf. Proposition 1 below) that any Dirac structure $D \subset E$ can be regarded as the graph of a map from $E_+ \to E_-$, which, by the above identifications, corresponds to a (point-wise) map $\mathcal{O}$ from TM to itself. Let us denote the E_+ and E_- decomposition of an element of E as $(v_1; v_2)$, where elements v_i may be regarded as vectors on M. Then any Dirac structure can be written as $D = \{(v; \mathcal{O}v) \in E, v \in TM\}$, where $\mathcal{O}$ is point-wise orthogonal w.r.t. the metric g. Obviously, $(v; \mathcal{O}v) = (1 + \mathcal{O})v \oplus (1 - \mathcal{O})v \in TM \oplus TM \cong T^*M \oplus TM = E$; thus e.g. the graph $D = \mathrm{graph}(\mathcal{P})$ of a bivector field $\mathcal{P}$, $D = \{\alpha \oplus \mathcal{P}(\alpha, \cdot), \alpha \in T^*M\} \subset E$ corresponds to the orthogonal operator

$$\mathcal{O} = \frac{1 - \mathcal{P}}{1 + \mathcal{P}} \quad \Leftrightarrow \quad \mathcal{P} = \frac{1 - \mathcal{O}}{1 + \mathcal{O}}\,. \tag{22}$$

Note that in a slight abuse of notation we did not distinguish between the bivector field $\mathcal{P} = \frac{1}{2}\mathcal{P}^{ij}\partial_i \wedge \partial_j \in \Gamma(\Lambda^2 TM)$, the canonically induced map from $T^*M \to TM$, $\alpha \mapsto \mathcal{P}(\alpha, \cdot)$, and the corresponding operator on TM using the isomorphism induced by g: $TM \ni v \mapsto \mathcal{P}(\mathrm{g}(v, \cdot), \cdot) \in TM$; in particular this implies that in an explicit matrix calculation using some local basis ∂_i in TM, with $\mathcal{O} = \mathcal{O}^i_j \partial_i \otimes \mathrm{d}X^j$, the matrix denoted by $\mathcal{P}$ in (22) is $\mathcal{P}_i{}^j \equiv \mathrm{g}_{ik}\mathcal{P}^{kj}$.

Obviously the Dirac structure of the G/G model corresponds to the choice Ad_g for $\mathcal{O}$ above and the first formula (16) is the specialization of (22) to this particular case. The transformation (22) is a Cayley map. Although any antisymmetric matrix $\mathcal{P}$ yields an orthogonal matrix $\mathcal{O}$ by this transformation, the reverse is not true. This is the advantage of using $\mathcal{O}$ over $\mathcal{P}$, as it works for any Dirac structure.[4] Certainly such as the bivector of WZ-Poisson structure has to satisfy an integrability condition, namely Eq. (11), which for $H = 0$ states that $\mathcal{P}$ defines a Poisson structure. There is a likewise condition to be satisfied by $\mathcal{O}^i_j(X)$ so that, more generally, $\mathcal{O}$ describes a Dirac structure. $\mathcal{O}$ corresponds to a Dirac structure iff $U := 1 - \mathcal{O}$ satisfies (cf. Proposition 2 below):

$$U^{\tilde{i}}{}_i U^{\tilde{j}}{}_{j;\tilde{i}} (1-U)_{\tilde{j}k} + cycl(i,j,k) = \tfrac{1}{2} H_{\tilde{i}\tilde{j}\tilde{k}} U^{\tilde{i}}{}_i U^{\tilde{j}}{}_j U^{\tilde{k}}{}_k \,. \tag{23}$$

Here the semicolon denotes the covariant derivative with respect to the Levi Civita connection of g. Locally it may be replaced by an ordinary partial derivative, if the auxiliary metric is chosen to be flat on some coordinate patch.

Having characterized D by $\mathcal{O} \in \Gamma(O(TM))$, we may parameterize $\mathcal{A} \in \Omega(\Sigma, X^*D)$ more explicitly by $a = a^i \partial_i \in \Omega(\Sigma, X^*TM)$ according to $\mathcal{A} = -(1+\mathcal{O})a \oplus -(1-\mathcal{O})a$.

Then the total action (20) can be rewritten in the form

$$\begin{aligned} S_{\mathrm{DSM}}[X,a] &= \frac{\alpha}{2} \int_\Sigma ||\mathrm{d}X + (1-\mathcal{O})a||^2 + \int_\Sigma \mathrm{g}(\mathrm{d}X \overset{\wedge}{,} (1+\mathcal{O})a) + \mathrm{g}(a \overset{\wedge}{,} \mathcal{O}a) + \int_{N_3} H \\ &\equiv \frac{\alpha}{2} \int_\Sigma (\mathrm{d}X^i + a^i - \mathcal{O}^i_k a^k) \wedge *(\mathrm{d}X^j + a^j - \mathcal{O}^j_m a^m)\, \mathrm{g}_{ij} \\ &\quad + \int_\Sigma \mathrm{d}X^i \wedge a^j\, (\mathrm{g}_{ij} + \mathcal{O}_{ij}) + \mathcal{O}_{ij} a^i \wedge a^j + \int_{N_3} H, \end{aligned} \tag{24}$$

where now X^i and a^i, local 0-forms and 1-forms on Σ, respectively, can be varied without any constraints and indices are lowered and raised by means of $\mathrm{g}_{ij}(X)$ and $\mathrm{g}^{ij}(X)$, respectively. We stress again that the g-dependence of the last line is ostensible only, whereas the α-dependent terms depend on it inherently.

The above presentation of $\mathcal{A}$ was suggested by the G/G model. In the rest of the paper we will however rather use the slightly more elegant parameterization $\mathcal{A} = (1 + \mathcal{O})\mathtt{a} \oplus (1 - \mathcal{O})\mathtt{a}$, resulting from $\mathtt{a} := -a$. In these variables the action (24) takes the form

$$S_{\mathrm{DSM}}[X,\mathtt{a}] = \frac{\alpha}{2} \int_\Sigma ||f||^2 + \int_\Sigma \mathrm{g}((1+\mathcal{O})\mathtt{a} \overset{\wedge}{,} \mathrm{d}X) + \mathrm{g}(\mathtt{a} \overset{\wedge}{,} \mathcal{O}\mathtt{a}) + \int_{N_3} H, \tag{25}$$

where we used (19) with

$$f \equiv \mathrm{d}X - V \equiv \mathrm{d}X - (1-\mathcal{O})\mathtt{a}\,. \tag{26}$$

In the following sections we will tie the above formulas to a more mathematical framework and, among others, analyze the field equations from this perspective. Readers more interested in applications for physics may also be content with consulting only

[4] This observation is due to the collaboration of A.K. and T.S. with A. Alekseev and elaborated further in [1]. We are also grateful to A. Weinstein for pointing out to us that the description of Dirac structures by means of sections of $O(TM)$ was used already in the original work [8].

the main results from the following sections, in particular Theorem 1 and Proposition 7, and then turn directly to the Hamiltonian analysis of the action in Sect. 6.

Maybe also noteworthy is the generalization Eq. (44) of the kinetic term introduced in Sect. 4 below. The modification of the old kinetic term uses a 2-form C on the target and is independent of any metric. Such a generalized kinetic term is suggested by the more mathematical considerations to follow, but will not be pursued any further within the present paper.

We close this section with a continuative remark: As mentioned above, for non-vanishing parameter α, the classical theory will turn out not to depend on this parameter. It is tempting to believe that this property can be verified also on the quantum level, a change in α corresponding to the addition of a BRS-exact term. In this context it may be interesting to regard the limit $\alpha \to \infty$, yielding localization to $f = 0$. In fact, one may expect localization of the path integral to all equations of motion, cf. [34, 13].

3. Dirac Structures

The purpose of this section is to provide readers with the mathematical background for the structures used to define the Dirac sigma model. We review some basic facts about Dirac structures, being maximally isotropic (Lagrangian) subbundles in an exact Courant algebroid, the restriction of the Courant bracket to which is closed. We describe an explicit isomorphism between the variety of all Lagrangian subbundles and the group of point-wise acting operators in the tangent bundles, orthogonal with respect to a fixed Riemann metric. We derive an obstruction for such operators to represent a Dirac structure, cf. Proposition 2 below.

A *Courant algebroid* [22, 8] is a vector bundle E equipped with a non-degenerate symmetric bilinear form $\langle , \rangle$, a bilinear operation $\circ$ on $\Gamma(E)$ (sometimes also denoted as a bracket $[\cdot, \cdot]$), and a bundle map $\rho : E \to TM$ satisfying the following properties:

1. The left Jacobi condition $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$,
2. Representation $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$,
3. Leibniz rule $e_1 \circ f e_2 = f e_1 \circ e_2 + L_{\rho(e_1)}(f) e_2$,
4. $e \circ e = \frac{1}{2}\mathcal{D}\langle e, e\rangle$,
5. Ad-invariance $\rho(e_1)\langle e_2, e_3\rangle = \langle e_1 \circ e_2, e_3\rangle + \langle e_2, e_1 \circ e_3\rangle$,

where $\mathcal{D}$ is defined as $\mathcal{D} : C^\infty(M) \stackrel{\mathrm{d}}{\to} \Omega^1(M) \stackrel{\rho^*}{\to} E^* \simeq E$. Properties 2 and 3 can be shown to follow from the other three properties, which thus may serve as axioms (cf. e.g. [19]). A Courant algebroid is called *exact* [28], if the following sequence is exact:

$$0 \to T^*M \stackrel{\rho^*}{\to} E \stackrel{\rho}{\to} TM \to 0. \tag{27}$$

A *Dirac structure D* in an exact Courant algebroid is a maximally isotropic (or Lagrangian) subbundle with respect to the scalar product, which is closed under the product. A Dirac structure is always a particular Lie algebroid: By definition a Lie algebroid is a vector bundle $F \to M$ together with a bundle map $\rho\colon F \to TM$ and an antisymmetric product (bracket) between its sections satisfying the first three properties in the list above (where again the second property can be derived from the other two). In particular, the product, often also denoted as a bracket, $e_1 \circ e_2 :\equiv [e_1, e_2]$, defines an infinite dimensional Lie algebra structure on $\Gamma(F)$. Due to the isotropy of D, the induced product (bracket) becomes skew-symmetric and obviously D is a Lie algebroid.

From now on we will only consider exact Courant algebroids (27). Let us choose a "connection" on E, i.e. an isotropic splitting $\sigma : TM \to E$, $\rho \circ \sigma = \mathrm{id}$. The difference

$$\sigma(X) \circ \sigma(Y) - \sigma([X, Y]) = \rho^* H(X, Y) \tag{28}$$

is a pull-back of a $C^\infty(M)-$linear, completely skew-symmetric tensor $H \in \Omega^3(M)$, given by $H(X, Y, Z) = \langle \sigma(X) \circ \sigma(Y), \sigma(Z) \rangle$. From the above axioms one may deduce the "Bianchi identity": $\mathrm{d}H = 0$.

Once a connection is chosen, any other one differs by the graph of a $2-$form B. Its curvature is equal to $H + \mathrm{d}B$. Therefore the cohomology class $[H] \in H^3(M)$ is completely determined by the Courant algebroid [28].

Choosing a splitting with the curvature 3-form H, it is possible to identify the corresponding exact Courant algebroid with $T^*M \oplus TM$ and the scalar product with the natural one:

$$\langle \xi_1 + \theta_1, \xi_2 + \theta_2 \rangle = \theta_1(\xi_2) + \theta_2(\xi_1) \ , \tag{29}$$

where $\xi_i \in \Gamma(TM)$, $\theta_i \in \Omega^1(M)$. Finally, the multiplication law can be shown to take the form:

$$(\xi_1 + \theta_1) \circ (\xi_2 + \theta_2) = [\xi_1, \xi_2] + L_{\xi_1}\theta_2 - \imath_{\xi_2}\mathrm{d}\theta_1 + H(\xi_1, \xi_2, \cdot) \ , \tag{30}$$

where L_ξ and i_ξ denote the Lie derivative along a vector field ξ and contraction with ξ, respectively.

Let E be an exact Courant algebroid with a chosen splitting $E = T^*M \oplus TM$ and a vanishing 3-form curvature (this implies that the chacteristic 3-class of E is trivial). Then we have:

Example 1. Let D be a graph of a Poisson bivector field $\mathcal{P} \in \Gamma(\Lambda^2 TM)$ considered as a skew-symmetric map from T^*M to TM, then $D = \{\theta \oplus \mathcal{P}(\theta)\}$ is a Dirac subbundle and the projection from D to T^*M is bijective. Any Dirac subbundle of E with bijective projection to T^*M is a graph of a Poisson bivector field.

Example 2. Let D be a graph of a closed $2-$form $\omega \in \Omega^2(M)$ considered as a skew-symmetric map from TM to T^*M, then $D = \{\omega(v) \oplus v\}$ is a Dirac subbundle and the projection of D to TM is bijective. Any Dirac subbundle of E for which the projection to TM is bijective is a graph of a closed $2-$form.

If E has a non-trivial characteristic class $[H]$, one needs to replace the Poisson (presymplectic) structure by a corresponding WZ (or twisted) one, cf. [16, 29].

Now we describe all Lagrangian subbundles of an exact Courant algebroid E, which are not necessarily projectable, either to T^*M (for any splitting σ) or to TM.[5] Let us choose an arbitrary Riemannian metric g on M, which can be thought of as a non-degenerate symmetric map from TM to T^*M. The inverse of g is acting from T^*M to TM. We denote these actions as $\xi \mapsto \xi^*$ and $\theta \mapsto \theta^*$ for a vector field ξ and a 1-form θ, respectively. In some local coordinate chart it can be written as follows:

$$\xi^* = \left(\xi^i \partial_i\right)^* = \xi^i \mathrm{g}_{ij} \mathrm{d}x^j \ , \quad \theta^* = \left(\theta_i \mathrm{d}x^i\right)^* = \theta_i \mathrm{g}^{ij} \partial_j \ . \tag{31}$$

[5] At least most of the rest of the section seems to be original, taking into account, however, the previous footnote for what concerns Proposition 1 below.

Given a splitting σ, one can combine these maps to a bundle involution $\tau\colon E \to E$, $\theta \oplus \xi \mapsto \xi^* \oplus \theta^*$, with the obvious property $\tau^2 \equiv 1$. To simplify the notation, we will henceforth just write $\theta + \xi$ instead of $\theta \oplus \xi$, because the nature of θ and ξ anyway indicate the position in $E \cong T^*M \oplus TM$. The bundle E thus decomposes into $\pm 1-$eigenvalue parts, $E = E_+ \oplus E_-$, where $E_\pm := Ker(\tau \mp 1)$.

Proposition 1. *Any Lagrangian subbundle is a graph of an orthogonal map $E_+ \to E_-$, which can be identified with a section $\mathcal{O} \in \Gamma(O(TM))$.*

Proof. First, let us show that τ is symmetric with respect to $\langle , \rangle$. In fact, by the definition of the scalar product (29) and τ we have

$$\langle \tau(\xi_1 + \theta_1), \xi_2 + \theta_2 \rangle = \langle \theta_2, \theta_1^* \rangle + \langle \xi_1^*, \xi_2 \rangle = \mathrm{g}(\xi_1, \xi_2) + \mathrm{g}(\theta_1, \theta_2)\,.$$

Now it is easy to see that the restriction of $\langle , \rangle$ to E_+ (E_-) is a positive (negative) metric, respectively, and $\langle E_+, E_- \rangle \equiv 0$. Therefore we conclude that any Lagrangian subbundle has a trivial intersection with $E_\pm$. Hence the projection of D to E_+ is bijective which implies that D is a graph of some map $E_+ \to E_-$. Let us identify $E_\pm$ with TM by means of $\pm\rho$, then the map uniquely corresponds to an orthogonal transformation $\mathcal{O}$ of TM. More precisely, any section $u_\pm$ of $E_\pm$ can be uniquely represented as $\xi \pm \xi^*$ for some vector field ξ. Now the definition of $\mathcal{O}$ yields that any section of D has the form $(1-\mathcal{O})\xi + ((1+\mathcal{O})\xi)^*$ for a certain vector field ξ. Taking into account that $\langle , \rangle$ vanishes on D, we show that $\mathcal{O}$ is an orthogonal map:

$$\begin{aligned}&\langle (1-\mathcal{O})\xi + ((1+\mathcal{O})\xi)^*, (1-\mathcal{O})\xi' + ((1+\mathcal{O})\xi')^* \rangle \\ &\quad = \mathrm{g}((1-\mathcal{O})\xi, (1+\mathcal{O})\xi') + \mathrm{g}((1+\mathcal{O})\xi, (1-\mathcal{O})\xi') = 2(\xi, \xi') - 2\mathrm{g}(\mathcal{O}\xi, \mathcal{O}\xi') = 0.\end{aligned}$$

□

For the argumentation above, in particular for the fact that D has a trivial intersection with $E_\pm$, it was important that g is a metric of definite signature. Note, however, that even when g is an arbitrary pseudo-Riemannian metric, we obtain a maximally isotropic subbundle D from a graph in E_+, E_- of an pseudo-orthogonal operator $\mathcal{O}$; just not all such subbundles D can be characterized in this way. This is an important fact when we want to cover, e.g., the G/G-model for non-compact semi-simple Lie groups.

Locally not any Dirac structure D admits a splitting σ such that D corresponds to either a Poisson or a presymplectic structure. But even if it does so locally, there may be global obstructions for it to be WZ-Poisson or WZ-presymplectic. This can be shown by constructing characteristic classes associated to a given Dirac structure $D \subset E$ [1]. An example for such a Dirac structure with "non-trivial winding" is the following one:[6]

Example 3. Take $M = \mathrm{G}$ a Lie group whose Lie algebra $\mathfrak{g} = Lie\,\mathrm{G}$ is quadratic, with the non-degenerate ad-invariant scalar product denoted by $\langle \cdot, \cdot \rangle$.

Then the respective exact Courant algebroid $E = T^*\mathrm{G} \oplus T\mathrm{G}$ can be cast into the following form:

$$\begin{array}{rcl} E &=& \mathrm{G} \times (\mathfrak{g} \oplus \mathfrak{g}), \\ \rho(x,y) &=& x^R \equiv x\mathrm{g}, \\ \langle (x,y),(x',y') \rangle &=& \langle x, y' \rangle + \langle x', y \rangle, \\ (x,y) \circ (x',y') &=& (-[x,x'], [x,x'] - [x,y'] + [x',y]), \ \ \forall \text{const. sections } x, x', y, y' \end{array}.$$

[6] This example can be extracted directly from the previous section, cf. the text after formula (16)—we only changed from a right-invariant basis to a left-invariant one.

The cotangent bundle $T^*\mathrm{G}$ is embedded as follows: $\theta \mapsto (0, \theta^*\mathrm{g}^{-1})$, where θ^* is the vector field dual to the 1-form θ via the Killing metric on G which is left- and right-invariant. Note that for any left or right invariant vector field ξ one has $L_\xi(\theta)^* = (L_\xi\theta)^*$.

It is easy to see that the curvature H of the splitting (connection) $\sigma : \xi \to (\xi g^{-1}, 0)$ equals the Cartan 3-form

$$H(\xi_1, \xi_2, \xi_3) = \langle \xi_1 \mathrm{g}^{-1}, [\xi_2 \mathrm{g}^{-1}, \xi_3 \mathrm{g}^{-1}]\rangle ,$$

for $\xi_i \in \Gamma(TG)$. The natural Dirac structure, considered in Sect. 2, is determined by $\mathcal{O} = Ad_{\mathrm{g}}$. One can calculate the product of two section of this Dirac structure in the representation defined above (here x, y are constant sections of $\mathrm{G} \times \mathfrak{g}$):

$$((1 - \mathcal{O})x, (1 + \mathcal{O})x) \circ ((1 - \mathcal{O})y, (1 + \mathcal{O})y) = (-(1 - \mathcal{O})[x, y], -(1 + \mathcal{O})[x, y]) . \tag{32}$$

Certainly, closure on $\Gamma(D)$ of the induced product or bracket requires some additional property of the operator $\mathcal{O}$, generalizing e.g. the Jacobi identity of the Poisson bivector in Example 1.

Proposition 2. *A Lagrangian subbundle, represented by an orthogonal operator $\mathcal{O}$ as the set $D = \{(1 - \mathcal{O})\xi + ((1 + \mathcal{O})\xi)^*\}$, is a Dirac structure, iff the following property holds, where ∇ denotes the Levi-Civita connection on M and $\xi_i \in \Gamma(M, TM)$:*

$$\sum_{\sigma\in\mathbb{Z}_3} \mathrm{g}\left(\mathcal{O}^{-1}\nabla_{(1-\mathcal{O})\xi_{\sigma(1)}}(\mathcal{O})\, \xi_{\sigma(2)} \;,\; \xi_{\sigma(3)}\right) = \frac{1}{2} H\left((1-\mathcal{O})\xi_1, (1-\mathcal{O})\xi_2, (1-\mathcal{O})\xi_3\right) . \tag{33}$$

Proof. First, we rewrite the multiplication law in terms of the Levi-Civita connection:

$$x_1 \circ x_2 = \nabla_{\rho(x_1)} x_2 - \nabla_{\rho(x_2)} x_1 + \langle \nabla x_1, x_2\rangle + H(\rho(x_1), \rho(x_2), \cdot) , \tag{34}$$

where $x, y \in E$, ∇x is thought of as a 1-form taking values in E, and hence $\langle \nabla x_1, x_2\rangle$ is in $\Omega^1(M) \subset E$.

Let us take $x_i \in \Gamma(M, D)$, $i = 1, 2, 3$, written in the form

$$x_i = (1 - \mathcal{O})\xi_i + ((1 + \mathcal{O})\xi_i)^* . \tag{35}$$

Note that $\rho(x_i) = (1 - \mathcal{O})\xi_i$. Using (34), we derive the product $x_1 \circ x_2$ and the 3-product $\langle x_1 \circ x_2, x_3\rangle$, which is a $C^\infty(M)$−linear form vanishing if and only if D is closed with respect to the Courant multiplication,

$$\begin{aligned} x_1 \circ x_2 = &(1-\mathcal{O}) \left(\nabla_{\rho(x_1)}\xi_2 - \nabla_{\rho(x_2)}\xi_1\right) + \left((1+\mathcal{O}) \left(\nabla_{\rho(x_1)}\xi_2 - \nabla_{\rho(x_2)}\xi_1\right)\right)^* \\ &+ H(\rho(x_1), \rho(x_2), \cdot) - \nabla_{\rho(x_1)}(\mathcal{O})\xi_2 + \nabla_{\rho(x_2)}(\mathcal{O})\xi_1 \\ &+ \left(\nabla_{\rho(x_1)}(\mathcal{O})\xi_2 - \nabla_{\rho(x_2)}(\mathcal{O})\xi_1\right)^* - 2\mathrm{g}\left(\mathcal{O}^{-1}\nabla(\mathcal{O})\xi_1, \xi_2\right) . \end{aligned} \tag{36}$$

In the above we used that the Levi-Civita connection commutes with τ, i.e. $\nabla_\xi(\eta^*) = \left(\nabla_\xi\eta\right)^*$. Apparently, the sum of the first and second terms in (36) belongs to the same maximally isotropic subbundle, therefore its product with x_3 vanishes, and $\langle x_1 \circ x_2, x_3\rangle = (I) + (II) + (III)$, where

$$\begin{aligned} (I) &= \langle -\nabla_{\rho(x_1)}(\mathcal{O})\xi_2 + \left(\nabla_{\rho(x_1)}(\mathcal{O})\xi_2\right)^* , (1-\mathcal{O})\xi_3 + ((1+\mathcal{O})\xi_3)^*\rangle - (1 \leftrightarrow 2) \\ &= \mathrm{g}\left(-\nabla_{\rho(x_1)}(\mathcal{O})\xi_2, (1+\mathcal{O})\xi_3\right) + \mathrm{g}\left(\nabla_{\rho(x_1)}(\mathcal{O})\xi_2, (1-\mathcal{O})\xi_3\right) - (1 \leftrightarrow 2) \\ &= -2\mathrm{g}\left(\mathcal{O}^{-1}\nabla_{\rho(x_1)}(\mathcal{O})\xi_2 + \mathcal{O}^{-1}\nabla_{\rho(x_2)}(\mathcal{O})\xi_3, \xi_1\right), \end{aligned}$$

and

$$(II) = \langle -2\mathrm{g}\left(\mathcal{O}^{-1}\nabla(\mathcal{O})\xi_1, \xi_2\right), x_3\rangle = -2\mathrm{g}\left(\mathcal{O}^{-1}\nabla_{\rho(x_3)}(\mathcal{O})\xi_1, \xi_2\right) ,$$
$$(III) = H\left((1-\mathcal{O})\xi_{\sigma(1)}, (1-\mathcal{O})\xi_{\sigma(2)}, (1-\mathcal{O})\xi_{\sigma(3)}\right) .$$

In the formulas above $(1 \leftrightarrow 2)$ denotes the permutation of the first two indices and $\mathbb{Z}_3$ is the group of cyclic permutations of order 3. We also used that the orthogonality of $\mathcal{O}$ implies that $\mathcal{O}^{-1}\nabla(\mathcal{O})$ is a skew-symmetric operator with respect to the metric g, i.e. $\mathrm{g}(\mathcal{O}^{-1}\nabla(\mathcal{O})\eta_1, \eta_2) = -\mathrm{g}(\mathcal{O}^{-1}\nabla(\mathcal{O})\eta_2, \eta_1)$ holds for any couple of vector fields η_1, η_2. All in all we then obtain

$$\langle x_1 \circ x_2, x_3\rangle = -2\sum_{\sigma\in\mathbb{Z}_3} \mathrm{g}\left(\mathcal{O}^{-1}\nabla_{(1-\mathcal{O})\xi_{\sigma(1)}}(\mathcal{O}), \xi_{\sigma(2)}, \xi_{\sigma(3)}\right)$$
$$+H\left((1-\mathcal{O})\xi_1, (1-\mathcal{O})\xi_2, (1-\mathcal{O})\xi_3\right) , \qquad (37)$$

which implies (33). □

From the above proof we extract the following useful

Corollary 1. *Assume that the integrability condition (33) holds and that* $x_i \in \Gamma(D)$*, parameterized as in (35). Then their Courant product (36) can be written as*

$$x_1 \circ x_2 = (1-\mathcal{O})Q(\xi_1, \xi_2) + ((1+\mathcal{O})Q(\xi_1, \xi_2))^* , \qquad (38)$$

where

$$Q(\xi_1, \xi_2) = \nabla_{\rho(x_1)}\xi_2 - \nabla_{\rho(x_2)}\xi_1 + \left(\mathrm{g}\left(\xi_1, \mathcal{O}^{-1}\nabla(\mathcal{O})\xi_2\right)\right)^* + \frac{1}{2}H\left(\rho(x_1), \rho(x_2), \cdot\right)^* , \qquad (39)$$

and $\rho(x_i) \equiv (1-\mathcal{O})\xi_i$.

At the expense of introducing an auxiliary metric g on M, a Dirac structure can be described globally by $\mathcal{O} \in \Gamma(O(TM))$. The introduction of $\mathcal{O}$ permits also to identify D with TM (via Eq. (35)). The Courant bracket thus induces a Lie algebroid bracket on D. This in turn induces an unorthodox Lie algebroid structure on TM, where the bracket between two vector fields ξ, ξ' is given by $[\xi, \xi'] := Q(\xi, \xi')$, which defines a Lie algebroid structure on $F := TM$ with anchor $\rho\colon F \to TM, \xi \mapsto (1-\mathcal{O})\xi$. In a holonomic frame, the corresponding structure functions, $[\partial_i, \partial_j]_F = C^k_{ij}\partial_k$, are easily computed as

$$C^k_{ij} = (1-\mathcal{O})^m{}_i\Gamma^k_{mj} - (i \leftrightarrow j) + \mathcal{O}^m{}_j{}^{;k}\mathcal{O}_{mi} + \frac{1}{2}H_{mn}{}^k(1-\mathcal{O})^m_i(1-\mathcal{O})^n_j . \qquad (40)$$

For practical purposes it may be useful to know how the orthogonal operator $\mathcal{O}$ transforms when changing g:

Proposition 3. *Given a fixed splitting so that* $E = T^*M \oplus TM$*, the couples* $(\mathrm{g}, \mathcal{O})$ *and* $(\mathrm{g}, \widetilde{\mathcal{O}})$ *describe the same Dirac structure D on M iff*

$$\widetilde{\mathcal{O}} = \left[\mathcal{O} - 1 + \widetilde{\mathrm{g}}^{-1}\mathrm{g}(1+\mathcal{O})\right]\left[1 - \mathcal{O} + \widetilde{\mathrm{g}}^{-1}\mathrm{g}(1+\mathcal{O})\right]^{-1} . \qquad (41)$$

Proof. An arbitrary element $\omega \oplus v$ in D can be parameterized as $g(1+\mathcal{O})\xi \oplus (1-\mathcal{O})\xi$ for some $\xi \in TM$. Equating this to $\widetilde{g}(1+\widetilde{\mathcal{O}})\widetilde{\xi} \oplus (1-\widetilde{\mathcal{O}})\widetilde{\xi}$, it is elementary to derive $\widetilde{\xi} = \frac{1}{2}\left[1-\mathcal{O}+\widetilde{g}^{-1}g(1+\mathcal{O})\right]\xi$. Since both of these two parameterizations are one-to-one (see Proposition 1), the dependence above is invertible. Using this relation in equating $(1-\mathcal{O})\xi$ to $(1-\widetilde{\mathcal{O}})\widetilde{\xi}$ for all ξ, we prove the statement of the proposition. □

As a simple corollary one obtains the following

Lemma 1. *For any orthogonal operator $\mathcal{O}$ and positive or negative symmetric operator b, the operator $1-\mathcal{O}+b(1+\mathcal{O})$ is invertible.*

Both assumptions in the lemma refer to a definite metric (as this was assumed and necessary for an exhaustive description of Dirac structures—cf. the discussion following Proposition 1). For later use we conclude from this

Corollary 2. *The operator*

$$T_\alpha := 1+\mathcal{O}+\alpha(1-\mathcal{O})* \tag{42}$$

on $T^*\Sigma \otimes X^*TM$ is invertible. Here $*$ is the Hodge operator on $T^*\Sigma$; for Lorentzian signatures of h, $*^2 = \mathrm{id}$ and, by assumption, $\alpha \in \mathbb{R}\backslash 0$, for Euclidean signatures, $*^2 = -\mathrm{id}$ and $i\alpha \in \mathbb{R}\backslash 0$.

The statement above follows in an obvious way from Lemma 1, i.e. for definite metrics g. For pseudo-Riemannian metrics g, however, it in general becomes necessary to restrict α to a neighborhood of $\alpha = 1$ and $\alpha = i$ for Lorentzian and Euclidean signature of h, respectively.

4. Change of Splitting

The action S_{DSM} of the Dirac Sigma Model consists of two parts, the topological term (17) and the kinetic one (18). It was mentioned repeatedly that only the second contribution depends on the auxiliary metrics g and h. However, also the first part S_{top} (and in fact now only this part) depends on another auxiliary structure, namely the choice of the splitting. We will show in the present section that this dependence is rather mild: It can be compensated by a coordinate transformation on the field space, which is trivial on the classical solutions (cf. Proposition 5 below; the transformation is α-dependent, so it changes if also the kinetic term is taken into account).

There is also an interesting alternative: Recall that $h^{-1} \otimes X^*g$ was used as a symmetric pairing in $\Gamma(T^*\Sigma \otimes X^*TM)$ to define S_{kin}. If in addition we are given a 2-form C on M, we can also use $h^{-1}(\mathrm{vol}_\Sigma) \otimes X^*C$ for a symmetric pairing, where vol_Σ is the volume 2-form on Σ induced by h, and $h^{-1}(\mathrm{vol}_\Sigma)$ denotes the corresponding bivector resulting from raising indices by means of h. Using the sum of both (or, more precisely, an α-dependent linear combination of them) to define S_{kin}, cf. Eq. (44) below, a change of splitting can be compensated by a simple change of the new background field C.

From Sect. 3 one knows that a splitting in an exact Courant algebroid is governed by 2-forms B. Namely, assume that $\sigma : TM \to E$ is a splitting, then any other one sends a vector field ξ to $\sigma_B(\xi) := \sigma(\xi) + B(\xi, \cdot)$.

Proposition 4. *The DSM action transforms under a change of splitting $\sigma \to \sigma_B$ according to:*

$$S_{\mathrm{DSM}} \mapsto \widetilde{S}_{\mathrm{DSM}} := S_{\mathrm{DSM}} + \frac{1}{2}\int_\Sigma B_{ij}\, f^i \wedge f^j \,, \tag{43}$$

where $f^i \equiv \mathrm{d}X^i - V^i$.

Proof. In fact, the decomposition $\mathcal{A} = A + V$ is not unique and depends on the splitting. Changing the splitting by a 2-form B, we get a different decomposition: $\mathcal{A} = \widetilde{A} + V$, where $\widetilde{A} = A - B(V, \cdot)$. To argue for this we note that $V = \sigma\,(\rho(\mathcal{A}))$ is indeed invariant (in particular, S_{kin} is invariant), only A varies, hence after the change of splitting we obtain $\widetilde{A} = \mathcal{A} - \sigma_B\,(\rho(\mathcal{A})) = A - B(V, \cdot)$. Taking into account that the B-field influences H, $H \mapsto \widetilde{H} = H + dB$, we calculate:

$$\widetilde{S}_{top} = \int_\Sigma \widetilde{A}_i \wedge \mathrm{d}X^i - \frac{1}{2}\widetilde{A}_i \wedge V^i + \int_{N_3} \widetilde{H}$$
$$= S_{top} + \frac{1}{2}\int_\Sigma B(\mathrm{d}X \overset{\wedge}{,} \mathrm{d}X) - 2B(V \overset{\wedge}{,} \mathrm{d}X) + B(V \overset{\wedge}{,} V)\,,$$

which finally gives the required derivation (43). □

As mentioned above, if the kinetic term S_{kin} (8) is replaced by

$$S_{\text{kin}}^{\text{new}} := \frac{1}{2}\int_\Sigma \alpha\,\mathrm{g}(f \overset{\wedge}{,} *f) + C(f \overset{\wedge}{,} f) \tag{44}$$

for some auxiliary $C \in \Omega^2(M)$, then a change of splitting governed by the B-field merely leads to $C \mapsto C - B$ for this new background field.

Note that despite the fact that H and C change in a similar way w.r.t. a change of splitting, $H \mapsto H + \mathrm{d}B$, $C \mapsto C - B$, they enter the sigma model qualitatively in quite a different way: H is already *uniquely* given by the Courant algebroid and a chosen splitting, while C is on the same footing as g or h, which have to be chosen in addition. In what follows we will show that a change of splitting does not necessarily lead to a change in the background fields, but instead can also be compensated by a transformation of the field variables—at least infinitesimally.

Proposition 5. *Let* $\alpha \neq 0$ *and* B *be a "sufficiently small" 2-form. Then there exists a change of variables* $\bar{\mathsf{a}} := \mathsf{a} + \delta\mathsf{a}$ *such that* $S_{\text{DSM}}[X, \bar{\mathsf{a}}] \equiv S_{\text{DSM}}[X, \mathsf{a}] + \frac{1}{2}\int_\Sigma B(f \overset{\wedge}{,} f)$.

Clearly, $\delta\mathsf{a}$ needs to vanish for $f = 0$. We remark that this equation is one of the field equations, cf. Theorem 1 below, so that $\delta\mathsf{a}$ corresponds to an on-shell-trivial coordinate transformation (on field space).

Proof. We find that after the change of variables $\bar{\mathsf{a}} := \mathsf{a} + \delta\mathsf{a}$ one has

$$S_{\text{DSM}}[X, \bar{\mathsf{a}}] - S_{\text{DSM}}[X, \mathsf{a}] = \int_\Sigma \mathrm{g}(T_\alpha\,\delta\mathsf{a} \overset{\wedge}{,} f) + \frac{1}{2}\mathrm{g}(\delta\mathsf{a} \overset{\wedge}{,} R_\alpha\,\delta\mathsf{a})\,, \tag{45}$$

where T_α is given by Eq. (42) and

$$R_\alpha = \mathcal{O} - \mathcal{O}^{-1} + \alpha(2 - \mathcal{O} - \mathcal{O}^{-1})*\,. \tag{46}$$

Solving the equation $S_{\text{DSM}}[X, \bar{\mathsf{a}}] - S_{\text{DSM}}[X, \mathsf{a}] = \frac{1}{2}\int_\Sigma B(f \overset{\wedge}{,} f)$, we use the ansatz $\delta\mathsf{a} = T_\alpha^{-1} L f$ for some yet undetermined $L \in \Gamma(\Sigma, \mathrm{End}(T^*\Sigma \otimes X^*TM))$ (invertibility of T_α follows from Corollary 2). This yields the following equation for L:

$$L^*AL + L^* - L + B = 0\,, \tag{47}$$

where $A = T_\alpha^{-1*} R_\alpha^* T_\alpha^{-1}$ and B denotes the operator via the identification $B(a \overset{\wedge}{,} b) = \mathrm{g}(Ba \overset{\wedge}{,} b)$, i.e. the operator is obtained from the bilinear form by raising the second index. In the above the adjoint L^* of an operator L in $T^*\Sigma \otimes X^*TM$ is defined by means of the canonical pairing induced by g: $\mathrm{g}(a \overset{\wedge}{,} Lb) = \mathrm{g}(L^*a \overset{\wedge}{,} b)$.[7] For sufficiently small B, the above operator (or matrix) equation (47) has the following solution:

$$L = \left(\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n+1} (BA)^n \right) B\,, \tag{48}$$

where $\binom{\lambda}{n} \equiv \frac{\lambda(\lambda-1)\ldots(\lambda-n+1)}{n!}$. Note that the L above is anti-selfadjoint (antisymmetric), $L^* = -L$, so it solves the simplified equation $LAL + 2L - B = 0$. For the case that A has an inverse, the above solution can also be rewritten in the more transparent form $L = A^{-1}\left(\sqrt{1+AB} - 1\right)$. Since A can be seen to be bounded, the sum (48) converges for small enough B. □

For completeness we also display how $\mathcal{O}$ transforms under a change of splitting:

Proposition 6. *Given two splittings $\sigma, \widetilde{\sigma}$ of the exact Courant algebroid E, the couples $(\sigma, \mathcal{O})$ and $(\widetilde{\sigma}, \widetilde{\mathcal{O}})$ describe the same Dirac structure D on M, iff*

$$\widetilde{\mathcal{O}} = (2\mathcal{O} - B(1-\mathcal{O}))\,(2 - B(1-\mathcal{O}))^{-1}\ , \tag{49}$$

where B is defined as follows: $\widetilde{\sigma}(\xi) = \sigma(\xi) + (B\xi)^*$ for any vector field ξ.

Proof. Straightforward calculations similar to Proposition 3. □

5. Field Equations and Gauge Symmetries

In this section we compute the equations of motion of the Dirac sigma model (DSM) introduced in Sect. 2 and reinterpret them mathematically. In particular we will show that the collection $(X, \mathcal{A})$ of the fields of the DSM are solutions to the field equations if and only if they correspond to a morphism from $T\Sigma$ to the Dirac structure D, viewed as Lie algebroids. As a consequence they are also independent of the choice of metrics used to define the kinetic term S_{kin} of the model as well as of the splitting used to define S_{top}.

Definition 1 ([23, 5]). *A vector bundle morphism $\phi\colon E_1 \to E_2$ between two Lie algebroids with the anchor maps $\rho_i : E_i \to TM_i$ is a* **morphism of Lie algebroids***, iff the induced map $\Phi\colon \Gamma(\Lambda E_2^*) \to \Gamma(\Lambda E_1^*)$ is a chain map with respect to the canonical differentials* d_i*:*

$$\mathrm{d}_1\Phi - \Phi\mathrm{d}_2 = 0\,. \tag{50}$$

[7] For operators commuting with the Hodge dual operation $*$ (which applies to all operators appearing here), this coincides with the adjoint defined by the symmetric pairing induced by g and h: $\mathrm{g}(a \overset{\wedge}{,} {*Lb}) = \mathrm{g}(L^*a \overset{\wedge}{,} {*b})$. One may then verify e.g. $R_\alpha^* = -R_\alpha$, $*^* = -*$, $A^* = -A$, and $T_\alpha^* = \mathcal{O}^{-1}T_\alpha$. This notation is not to be confused with the isomorphism between TM and T^*M, extended to an involution $\tau\colon TM \oplus T^*M$, denoted by the same symbol, cf. Eq. (31).

First, notice that, fixing a base map $X\colon M_1 \to M_2$, any vector bundle morphism is uniquely determined by a section $a \in \Gamma(M_1, E_1^* \otimes X^* E_2)$. Hence the morphism property (50) should have a reformulation in terms of the couple (X, a). Second, it is easy to see that the property (50) is purely local, therefore it admits a description for any local frame.

Indeed, let $\{e_i\}$ be a local frame of the vector bundle E_2, $\{e^i\}$ be its dual, and $a^i := \Phi(e^i)$, then (50) is equivalent to the following system of equations:

$$\mathrm{d}_1 X - \rho_2(a) = 0, \tag{51}$$

$$\mathrm{d}_1 a^k + \frac{1}{2} C^k_{ij}\, a^i \wedge a^j = 0. \tag{52}$$

The first equation is covariant; it implies that the following commutative diagram holds true:

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \rho_1 \downarrow & & \rho_2 \downarrow \\ TM_1 & \xrightarrow{X_*} & TM_2 \end{array}$$

The second equation depends on the choice of frame. However, the additional contribution, which arises in (52) under a change of frame, is proportional to the $\mathrm{d}_1 X - \rho_2(a)$, hence it is of no effect, if Eq. (51) holds. For further details about this definition we refer to [5].

In our context $E_1 = T\Sigma$ and thus d_1 becomes the ordinary de Rham differential. The a above becomes $\mathcal{A} \in \Omega^1(\Sigma, X^* D)$, but can be identified with $\mathtt{a}$ due to

$$\mathcal{A} \equiv A + V = ((1 + \mathcal{O})\mathtt{a})^* + (1 - \mathcal{O})\mathtt{a}\,, \tag{53}$$

where $\mathtt{a} \in \Omega^1(\Sigma, X^* TM)$ is an unrestricted field. In these variables S_{DSM} has the form (25). The first morphism property, Eq. (51), then takes the form $f \equiv \mathrm{d}X - V = 0$.

Theorem 1. *Let $\alpha \neq 0$ (depending on the signatures of h and $\mathtt{g}$, possibly further restricted as specified after Eq. (24) or at the end of Sect. 3). Then the field equations of S_{DSM} have the form*

$$f \equiv \mathrm{d}X - (1 - \mathcal{O})\mathtt{a} = 0\,, \tag{54}$$

$$\nabla(\mathtt{a}) + \frac{1}{2}\mathtt{g}\left(\mathtt{a} \stackrel{\wedge}{,} \mathcal{O}^{-1}\nabla(\mathcal{O})\mathtt{a}\right)^* + \frac{1}{4} H((1 - \mathcal{O})\mathtt{a} \stackrel{\wedge}{,} (1 - \mathcal{O})\mathtt{a}, \cdot)^* = 0\,, \tag{55}$$

or, in the dependent (A, V) variables, $\mathrm{d}X = V$ and

$$\nabla A + \frac{1}{4}\mathtt{g}\left(V \stackrel{\wedge}{,} \mathcal{O}^{-1}\nabla(\mathcal{O})V\right) + \frac{1}{4}\langle A \stackrel{\wedge}{,} \mathcal{O}^{-1}\nabla(\mathcal{O})A^*\rangle + \frac{1}{2} H(V \stackrel{\wedge}{,} V \stackrel{\wedge}{,} \cdot) = 0\,. \tag{56}$$

The fields $(X, \mathcal{A})$ are a solution of the equations of motion, if and only if they induce a Lie algebroid morphism from $T\Sigma$ to D.

Corollary. *The classical solutions of the DSM do not depend on the choice of the coupling constant $\alpha \neq 0$ (in the permitted domain), or, more generally, on the choice of metrics $\mathtt{g}$ and h.*

Proof. Using that $\mathcal{O}$ is an orthogonal operator w.r.t. g, one computes in a straightforward generalization of (12) and (13)[8]

$$\mathcal{O}\left(\frac{\delta S_{\mathrm{DSM}}}{\delta \mathtt{a}}\right)^{*} = (1+\mathcal{O}+\alpha(1-\mathcal{O})*)\, f\,. \tag{57}$$

The term in the brackets is the operator (42), which, according to our assumption on α, is invertible; so, indeed $f=0$. Note, however, that in general $1+\mathcal{O}$ is invertible only if the Dirac structure corresponds to a graph of a bivector. Thus, *only* in the WZ-Poisson case one may drop the kinetic term altogether *if* one wants to keep the morphism property of the field equations.

We now turn to the X-variation of S_{DSM}. This is conceptually more subtle since $\mathtt{a}\in\Omega^1(\Sigma, X^*TM)$ depends implicitly on X as well. Thus to determine $\delta_X a$ we need a connection, since, heuristically, we are comparing sections in two different, but nearby bundles X_0^*TM and $(X_0+\delta X)^*TM$. (If we required e.g. $\delta_X \mathtt{a}^i=0$, then this would single out a particular holonomic frame, since a change of coordinates on M yields $\widetilde{\mathtt{a}}^i = M^i_j(X)\mathtt{a}^j$. In the following we develop an inherently covariant formalism that also produces covariant field equations.)

Let us denote the local basis in X^*TM dual to $\mathfrak{d}X^i$ in X^*T^*M by $\mathfrak{d}_i$. The notation $\mathfrak{d}X^i$ is used so as to distinguish it from d acting on the pull-back function X^*X^i, which we denote as usual by $\mathrm{d}X^i$. Then

$$\delta_X \mathfrak{d}_i = \Gamma^j_{ki}\delta X^k \mathfrak{d}_j\,, \tag{58}$$

where Γ^i_{jk} are coefficients of the Levi Civita connection ∇ of g. Also, we think of $\mathtt{a}^i$ to depend on both $X(x)$ and x (cf. also [5] for further details); correspondingly, $\delta_X \mathtt{a} = \left(\delta_X(\mathtt{a}^i) + \Gamma^i_{kj}\,\mathtt{a}^j\delta X^k\right)\mathfrak{d}_i$. Note that certainly $\delta_X \mathrm{d} = \mathrm{d}\delta_X$, where d denotes the de Rham operator. However, this does not apply for $\mathrm{d}X$ used above, which is the section in $T^*\Sigma\otimes X^*TM$ corresponding to the bundle map $X_*\colon T\Sigma\to TM$, $\mathrm{d}X = \mathrm{d}X^i\otimes\mathfrak{d}_i$; so in $\mathrm{d}X$, d does not denote an operator. Here one finds in analogy to $\delta_X a$,

$$\delta_X(\mathrm{d}X) = \left(\mathrm{d}(\delta X^i) + \Gamma^i_{kj}\mathrm{d}X^j\delta X^k\right)\,\mathfrak{d}_i = \triangledown(\delta X)\,, \tag{59}$$

where $\triangledown$ is the pull-back of the Levi-Civita connection ∇ to $X^*(TM)$ and in the last step torsion freeness of ∇, $\Gamma^i_{kj}=\Gamma^i_{jk}$, was used. The above covariant form of the variation (58) implies in particular that $\delta_X \mathrm{g}=0$, when g is viewed as an element in $\Gamma(\Sigma, X^*T^*M^{\otimes 2})$—as it appears in the action functional (24), so we will be permitted to use $\delta_X \mathrm{g}(a \stackrel{\wedge}{,} b) = \mathrm{g}(\delta_X a \stackrel{\wedge}{,} b) + \mathrm{g}(a \stackrel{\wedge}{,} \delta_X b)$ below.

With the above machinery at hand, the variation w.r.t. X is rather straightforward again. By construction it will produce a covariant form of the respective field equations. Since we already know that $f=0$ holds true, moreover, we will be permitted to drop all terms below which are proportional to f. Correspondingly, S_{kin} can be dropped for the

[8] Recall that $a=-\mathtt{a}$ and $\mathcal{O}=\mathrm{Ad}_g$ in the G/G model. The variational derivative is defined according to $\delta_{\mathtt{a}}S_{\mathrm{DSM}} = \int_\Sigma \langle\delta\mathtt{a} \stackrel{\wedge}{,} \frac{\delta S_{\mathrm{DSM}}}{\delta\mathtt{a}}\rangle \equiv \int_\Sigma \mathrm{g}\left(\delta\mathtt{a} \stackrel{\wedge}{,} \left(\frac{\delta S_{\mathrm{DSM}}}{\delta\mathtt{a}}\right)^*\right)$. Alternatively, one may infer this relation also from Eq. (45), keeping only the terms quadratic in $\delta\mathtt{a}$: since $\mathrm{g}(T_\alpha\,\delta\mathtt{a} \stackrel{\wedge}{,} f) = \mathrm{g}(\delta\mathtt{a} \stackrel{\wedge}{,} \mathcal{O}^{-1}T_\alpha f)$, cf. footnote.

calculation of $\delta_X S_{\text{DSM}} = 0$, since S_{kin} is quadratic in f. By convention, we will denote equalities up to $f = 0$ in what follows by $\approx$; so, in particular

$$\frac{\delta S_{\text{DSM}}}{\delta X} \approx \frac{\delta S_{\text{top}}}{\delta X}\,. \tag{60}$$

Also, we may drop all terms containing $\delta_X \mathsf{a}$, since on behalf of (57), they will be proportional to f again, with or without the kinetic term included (corresponding only to $\alpha \neq 0$ and $\alpha = 0$, respectively, in formula (57)). Thus,

$$\delta_X S_{\text{top}} \approx \int_\Sigma \mathsf{g}\left((\delta_X \mathcal{O})\mathsf{a} \stackrel{\wedge}{,} \mathrm{d}X\right) + \mathsf{g}\left((1+\mathcal{O})\mathsf{a} \stackrel{\wedge}{,} \nabla \delta X\right)$$
$$+\mathsf{g}\left(\mathsf{a} \stackrel{\wedge}{,} (\delta_X \mathcal{O})\mathsf{a}\right) + \frac{1}{2} H_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j\, \delta X^k\,, \tag{61}$$

where we already made use of Eq. (59). In the second term we perform a partial integration (dropping eventual boundary terms) and observe that

$$\nabla\left((1+\mathcal{O})\mathsf{a}\right) \approx 2\nabla \mathsf{a}\,, \tag{62}$$

which follows from $2\mathsf{a} \approx (1+\mathcal{O})\mathsf{a} + \mathrm{d}X$ and $\nabla(\mathrm{d}X) = -\mathrm{d}X^i \wedge \mathrm{d}X^j\, \Gamma^k_{ji} \otimes \partial_k \equiv 0$. Replacing $\mathrm{d}X$ by $\mathsf{a} - \mathcal{O}\mathsf{a}$ in the first term, we then obtain

$$\delta_X S_{\text{top}} \approx \int_\Sigma \mathsf{g}\left(\mathsf{a} \stackrel{\wedge}{,} (\mathcal{O}^{-1}\delta_X \mathcal{O})\mathsf{a}\right) + \mathsf{g}\left(2\nabla \mathsf{a} \stackrel{\wedge}{,} \delta X\right) + \frac{1}{2} H_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j\, \delta X^k\,, \tag{63}$$

where the first and the third term in (61) combined into the first term above. This proves Eq. (55). Equivalence with Eq. (56) is established easily as follows: For the first term we read Eq. (62) from right to left and use $A^* = (1+\mathcal{O})\mathsf{a}$. For the second term of (55) we replace a by $\frac{1}{2}(A^* + V)$ and utilize the antisymmetry of $\mathcal{O}^{-1}\nabla\mathcal{O}$ to cancel off-diagonal terms. For the third one we use $V = (1-\mathcal{O})\mathsf{a}$.

This leaves us with proving the equivalence of (55) to the second morphism property (52), specialized to the present setting, where, again, $f = 0$ may be used freely. So, we need to show that (55) can be replaced by $\mathrm{d}\mathsf{a}^i + \frac{1}{2} C^i_{jk} \mathsf{a}^j \wedge \mathsf{a}^k \approx 0$, where the structure functions are given by Eq. (40). Since $C^i_{jk} \mathsf{a}^j \wedge \mathsf{a}^k \otimes \partial_i \equiv Q(\partial_j, \partial_k)\, \mathsf{a}^j \wedge \mathsf{a}^k$, most of the terms in (55) are identified easily, and it only remains to show that

$$\nabla \mathsf{a} \equiv \left(\mathrm{d}\mathsf{a}^i + \Gamma^i_{jk} \mathrm{d}X^j \wedge \mathsf{a}^k\right) \otimes \partial_i \approx \left(\mathrm{d}\mathsf{a}^i + \mathsf{a}^j \wedge \mathsf{a}^k (1-\mathcal{O})^m{}_j \Gamma^i_{mk}\right) \otimes \partial_i\,, \tag{64}$$

which is an obvious identity. □

Having established that the field equations enforce a Lie algebroid morphism $T\Sigma \to D$, it is natural to expect that on solutions the gauge symmetries correspond to a homotopy of such morphisms [5]. This is indeed the case. For the gauge invariance of an action functional, however, an off-shell (and preferably global) definition of the symmetries is needed. This is a more subtle question. A derivation of the gauge symmetries, including a specialization to known examples, requires some more extended discussion for a sufficiently clear presentation. Moreover, the discussion fits into a more general framework, following the considerations in [5], and thus will be presented elsewhere [20]. In the present paper we only provide the result of such an analysis:[9]

[9] Note, however, that e.g. the fact that the model contains no propagating degrees of freedom (is "topological") follows also from the self-contained Hamiltonian analysis in the subsequent section.

Proposition 7. *For nonvanishing α the infinitesimal gauge symmetries of $S_{\rm DSM}$ can be expressed in the following form*

$$\delta_\epsilon X = (1-\mathcal{O})\epsilon \ , \tag{65}$$

$$\begin{aligned}\delta_\epsilon \mathsf{a} &= \nabla(\epsilon) - \mathsf{g}\left(\mathcal{O}^{-1}\nabla(\mathcal{O})\mathsf{a}, \epsilon\right)^* + \frac{1}{2}H((1-\mathcal{O})\mathsf{a}, (1-\mathcal{O})\epsilon, \cdot)^* \\ &\quad + T_\alpha^{-1}\left(\frac{1}{2}H(f, (1-\mathcal{O})\epsilon, \cdot)^* + (1-\alpha *)\nabla_f(\mathcal{O})\epsilon + Mf\right) \ ,\end{aligned} \tag{66}$$

where $\epsilon \in \Gamma(\Sigma, X^*TM)$ and $M = M^* \in \Gamma(\mathrm{End}(T^*\Sigma \otimes X^*TM))$ may be chosen freely.

The operator M above parametrizes trivial gauge symmetries. In the G/G and in the Poisson case the above symmetries reproduce the known ones for $\alpha = 1$ and $\alpha = 0$, respectively. In general, however, the inverse of T_α is defined only for nonvanishing α, cf. Corollary 2.

6. Hamiltonian Formulation

In this section we derive the constraints of the DSM. For simplicity we restrict ourselves to closed strings, $\Sigma \cong S^1 \times \mathbb{R} \ni (\sigma, \tau)$. Here $\sigma \sim \sigma + 2\pi$ is the "spatial" variable around the circle S^1 (along the closed string) and τ is the "time" variable governing the Hamiltonian evolution.

The discussion will be carried out for more general actions in fact: We may regard any action of the form of $S_{\rm DSM}$, where D is required to be a maximally isotropic (but possibly non-involutive) subbundle of E; in other words for the present purpose we will consider any action of the form (24) for *any* orthogonal (X-dependent) matrix $\mathcal{O}^i_j$. Generalizing an old fact for PSMs [25], the corresponding constraints are "first class" (define a coisotropic submanifold in the phase space), iff D is a Dirac structure (i.e. iff the matrices $\mathcal{O}^i_j$ satisfy the integrability conditions (23) found above).

Let ∂ denote the derivative with respect to σ (the τ-derivative will be denoted by an overdot below) and let δ be the exterior differential on phase space. Then we have

Theorem 2. *For $\alpha \neq 0$ and $\Sigma \cong S^1 \times \mathbb{R}$, the phase space of S_{DSM}, D maximally isotropic in $E = T^*M \oplus TM$, may be identified with the cotangent bundle to the loop space in M with the symplectic form twisted by the closed 3-form H,*

$$\Omega = \oint_{S^1} \delta X^i(\sigma) \wedge \delta p_i(\sigma) d\sigma - \tfrac{1}{2}\oint_{S^1} H_{ijk}(X(\sigma))\, \partial X^i(\sigma)\, \delta X^j(\sigma) \wedge \delta X^k(\sigma)\, d\sigma \ , \tag{67}$$

subject to the constraint $J_{\omega,v}(\sigma) = v^i(\sigma, X(\sigma))p_i(\sigma) + \omega_i(\sigma, X(\sigma))\partial X^i(\sigma) = 0$, for any choice of $\omega \oplus v \in C^\infty(S^1) \otimes \Gamma(X^*D)$, or, in the description of D by means of $\mathcal{O} \in \Gamma(O(TM))$,

$$J \equiv (\mathcal{O}+1)\partial X + (\mathcal{O}-1)p = 0 \, . \tag{68}$$

The constraints are of the first class, if and only if D is a Dirac structure.

For Dirac structures D that may be written as the graph of a bivector $\mathcal{P}$ (for the splitting chosen—cf. Proposition 1 and Example 1), $1+\mathcal{O}$ is invertible; then obviously (68) can be rewritten as $\partial X - \mathcal{P}p = 0$ or $\partial X^i + \mathcal{P}^{ij} p_j = 0$ (cf. Eq. (22) and the text about notation following this equation!), which agrees with the well-known expression of the constraints in the WZ-Poisson sigma model [16].

Proof. To derive the Hamiltonian structure we follow the shortcut version of Diracs procedure advocated in [9]. For simplicity we first drop the WZ-term, manipulating $\int_\Sigma \mathcal{L} \mathrm{d}\sigma \wedge \mathrm{d}\tau := S_{\mathrm{DSM}} - \int_{N_3} H$ in a first step. With $\mathrm{d}x^- \wedge \mathrm{d}x^+ = -2\mathrm{d}\sigma \wedge \mathrm{d}\tau$ we obtain from S_{DSM} by a straightforward calculation

$$\begin{aligned}\mathcal{L}[X, \mathcal{A}_\pm] = &-\frac{\alpha}{2}\dot{X}^2 + \tfrac{1}{2}(A_+ + \alpha V_+ - A_- + \alpha V_-)\dot{X} \\ &+\frac{\alpha}{2}\partial X^2 + \tfrac{1}{2}(-A_+ - \alpha V_+ - A_- + \alpha V_-)\partial X \\ &-\tfrac{1}{2}V_-(A_+ + \alpha V_+) + \frac{1}{4}(A_+ V_- + A_- V_+)\,, \end{aligned} \tag{69}$$

where appropriate target index contraction is understood [canonically, V, $\dot{X}$, and ∂X carry an upper target-index, and $A_\pm$ (as well as p introduced below) a lower one; but all indices may be raised and lowered by means of the target metric g]. For what concerns the determination of momenta, $\alpha \neq 0$ is qualitatively quite different from $\alpha = 0$. Restricting to the first case for the proof of the present theorem, we may now employ the following substitution to introduce a new momentum field p:

$$-\frac{\alpha}{2}\dot{X}^2 + \beta\dot{X} \quad \tilde{\rightarrow} \quad p\dot{X} + \frac{1}{2\alpha}(p-\beta)^2\,, \tag{70}$$

which results from $-\frac{\alpha}{2}\dot{X}^2 \,\tilde{\rightarrow}\, p\dot{X} + \frac{1}{2\alpha}p^2$ after shifting p to $p-\beta$. Within an action functional any such two expressions—for arbitrary C-numbers $\alpha \neq 0$ and possibly field dependent functions β—are equivalent, classically (eliminate p by its field equations) or on the quantum level (Gaussian path integration over p). Applying this to the first line in (69) with $\beta = \frac{1}{2}(A_+ + \alpha V_+ - A_- + \alpha V_-)$ and noting that the last bracket in the third line vanishes due to $\mathcal{A}_\pm \in D$ and the isotropy condition posed on D, we obtain $\mathcal{L} \,\tilde{\rightarrow}\, \mathcal{L}_{\mathrm{Ham}}$ with

$$\begin{aligned}\mathcal{L}_{\mathrm{Ham}}[X, \mathcal{A}_\pm, p] = &\, p\dot{X} + \frac{1}{2\alpha}\left(p - \tfrac{1}{2}A_+ - \tfrac{1}{2}\alpha V_+ + \tfrac{1}{2}A_- - \tfrac{1}{2}\alpha V_-\right)^2 \\ &+\frac{\alpha}{2}\partial X^2 + \tfrac{1}{2}(-A_+ - \alpha V_+ - A_- + \alpha V_-)\partial X \\ &-\tfrac{1}{2}V_-(A_+ + \alpha V_+)\,. \end{aligned} \tag{71}$$

It is straightforward to check that the above terms can be reassembled such that

$$\begin{aligned}\mathcal{L}_{\mathrm{Ham}}[X, \mathcal{A}_\pm, p] = &\, p\dot{X} - V_- p - A_-\partial X \\ &+\frac{1}{8\alpha}\left[A_+ + \alpha V_+ - (A_- + \alpha V_-) - 2(p + \alpha\partial X)\right]^2 \\ &-\tfrac{1}{2}A_- V_- - p\partial X\,. \end{aligned} \tag{72}$$

We now want to show that the last two lines may be dropped in this expression. Here we have to be careful to take into account that A and V are in general not independent fields, but subject to the restriction that their collection $\mathcal{A} = A \oplus V$ lies in the isotropic subbundle D. First we note that $A_- V_- \equiv \frac{1}{2}\langle \mathcal{A}_-, \mathcal{A}_-\rangle = 0$ due to $\mathcal{A}_- \in D$. Next, with a shift $\mathcal{A}_+ \to \widetilde{\mathcal{A}}_+ := \mathcal{A}_+ + \mathcal{A}_-$, the $\mathcal{A}_-$-part drops out in the second line; this is particularly obvious in terms of independent fields $\mathtt{a}_\pm$, where the term in the square brackets takes the form $[(1+\mathcal{O}) + \alpha(1-\mathcal{O})]\,(\mathtt{a}_+ - \mathtt{a}_-) - 2(p + \alpha\partial X)$. After this shift, $\mathcal{A}_-$ enters the action only linearly anymore, and thus plays the role of a Lagrange multiplier. This already shows the appearance of the constraints $J_{\omega,v} = 0$. Parameterizing $\omega \oplus v \in D$ as $(1+\mathcal{O})\lambda \oplus (1-\mathcal{O})\lambda$, one obtains $J_{\omega,v} = \mathrm{g}(\mathcal{O}\lambda, (\mathcal{O}+1)\partial X + (\mathcal{O}-1)p)$, g being the Riemann metric, which vanishes for any λ (an unconstrained Lagrange multiplier field), iff (68) holds true.

To show that the remaining dependence of the lower two lines in (72) on $p \oplus \partial X \in E$ can be eliminated (by a further shift in the fields), is seen most easily in a path-integral type of argument[10]: Integrating over $\mathcal{A}_-$ (i.e. taking the path integral over λ), one obtains a delta function that constrains $p \oplus \partial X$ to lie in the Dirac structure D; correspondingly, the term $p\partial X$ gives no contribution since D is isotropic, and $-2(p + \alpha\partial X)$ can be absorbed into $\widetilde{A}_+ + \alpha\widetilde{V}_+$ by a further redefinition of $\widetilde{\mathcal{A}}_+$ into $\bar{\mathcal{A}}_+$. After these manipulations the last two lines reduce to $\frac{1}{8\alpha}\left([(1+\mathcal{O}) + \alpha(1-\mathcal{O})]\,\bar{\mathtt{a}}_+\right)^2$. This is the only dependence of the resulting action on $\bar{\mathtt{a}}_+$, which thus may be put to zero as well.[11]

There also exists an argument on the purely classical level for the above consideration: Denote $p \oplus \partial X$, taking values in E, by ψ, and $\mathcal{A}_-$ and $\widetilde{\mathcal{A}}_+ \equiv \mathcal{A}_+ + \mathcal{A}_-$ by λ_D and μ_D, respectively (the index D so as to stress the restriction to the subbundle $D < E$). Then $\mathcal{L}_{\mathrm{Ham}} = \mathcal{L}_{\mathrm{Ham}}[X, p, \lambda_D, \mu_D] = p\dot{X} - \langle \lambda_D, \psi\rangle + f_1(\mu_D - \psi) + f_2(\psi)$, where f_1 and f_2 are polynomial functions to be read off from (72) and, as always, $\langle\cdot,\cdot\rangle$ denotes the fiber metric in E. Next we observe that $\tau(D) = \{(v; -\mathcal{O}v)\} \cong D^*$ has a trivial intersection with $D = \{(v; \mathcal{O}v)\}$ and $E = D \oplus \tau(D)$; note that $\tau(D)$ is also isotropic by construction, but in general will not be a Dirac structure (cf. Eq. (33)); as indicated already by the notation, it can be identified with the dual D^* of D by means of $\langle\cdot,\cdot\rangle$. Thus ψ can be decomposed uniquely into components $\psi_D \in D$ and $\psi_D^* \in D^*$, $\psi = \psi_D + \psi_D^*$. With $\widetilde{\mu}_D := \mu_D - \psi_D$ the action takes the form $\mathcal{L}_{\mathrm{Ham}} = p\dot{X} - \langle \lambda_D, \psi_D^*\rangle + f_1(\widetilde{\mu}_D - \psi_D^*) + f_2(\psi)$, since for vanishing ψ_D^* the last two contributions reduce to $f_1(\widetilde{\mu}_D)$. As a consequence there exists $F(\psi, \widetilde{\mu}_D)$ with values in E such that $f_1(\widetilde{\mu}_D - \psi_D^*) + f_2(\psi) = \langle F, \psi_D^*\rangle$. With $F = F_D + F_D^*$ and due to the isotropy of D^* we then obtain $\mathcal{L}_{\mathrm{Ham}} = \mathcal{L}_{\mathrm{Ham}}(X, p, \mu_D, \widetilde{\lambda}_D) = p\dot{X} - \langle \widetilde{\lambda}_D, \psi_D^*\rangle + f_1(\widetilde{\mu}_D)$, where $\widetilde{\lambda}_D := \lambda_D - F_D$ has been introduced and f_1 is a quadratic function in its argument. As before we thus may drop $f_1(\widetilde{\mu}_D)$ (cf. also footnote 11), obtaining

$$\mathcal{L}_{\mathrm{Ham}} = \mathcal{L}_{\mathrm{Ham}}[X, p, \mathcal{A}_+, \mathcal{A}_-] \stackrel{\sim}{\to} \mathcal{L}_{\mathrm{Ham}}[X, p, \widetilde{\lambda}] = p\dot{X} - \mathrm{g}(\widetilde{\lambda}, J) \tag{73}$$

for some unconstrained Lagrange multiplier field $\widetilde{\lambda} \in TM$.

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[10] A purely Lagrangian argument will be provided in the subsequent paragraph.

[11] Note that on behalf of the permitted values for α the matrix $\alpha + 1 + (\alpha - 1)\mathcal{O}$ is non-degenerate—due to Lemma 1 or Corollary 2. So the above statement follows from the field equations of $\bar{a}_+$, and using $\bar{a}_+ = 0$ in the action is a permitted step in the procedure of [9]. However, even if g has indefinite signature and $\alpha \neq 0$ is chosen such that the above quadratic form for $\bar{\mathtt{a}}_+$ is degenerate, this contribution can be dropped, since then the action does not depend on directions of $\bar{\mathtt{a}}_+$ in the kernel of the matrix, so they also give no contribution to the action (alternatively, $\bar{\mathtt{a}}_+ = 0$ may be viewed as a gauge fixation for those directions then). We remark parenthetically that the rank of the matrix may depend on $X \in M$ in this case, but dropping the contribution to the action in question obviously is the right step.

Noting that the addition of the Wess-Zumino term only contributes to the symplectic form as specified in (67), we thus proved the main part of the theorem.

The statement about the first class property follows from specializing the results of [2], where a Hamiltonian system with symplectic form (67) and currents $J_{\omega,v}$ for an arbitrary subset of elements $\omega \oplus v \in E$ was considered. □

The constraint algebra of $J_{\omega,v} = 0$ in the above theorem is an example of the more general current algebra corresponding to an exact Courant algebroid E found in [2]—the first class property is tantamount to requiring a closed constraint or current algebra without anomalies. On the other hand, apparently the action S_{DSM} provides a covariant action functional that produces the above mentioned currents (as constraints or symmetry generators) for the case of an arbitrary Dirac structure. As shown above this is even true if D is maximally isotropic but possibly not a Dirac structure. It is an interesting open problem to consider the Hamiltonian structure of the action (20) for a non-isotropic choice of D (does the more general statement hold true that the functional S_{DSM} defines a topological theory iff $D \subset E$ is a Dirac structure?) or likewise to provide some other covariant action functional producing constraints of the form considered in [2] for arbitrary $D < E$.

We already observed above that the discussion of the Hamiltonian structure changes (and in fact becomes somewhat more intricate) for the case of vanishing coupling constant α. E.g. substitutions such as in Eq. (70) are illegitimate in this limit. Moreover, as observed already in previous sections of the paper, the kinetic term S_{kin} is even necessary in general to guarantee the morphism property of the field equations (it becomes superfluous only when D is the graph of a bivector); in fact, the number of independent field equations may even change from $\alpha \neq 0$ to $\alpha = 0$ (with the extreme case of $D = TM$, where for $\alpha = 0$ one obtains no equations at all). Thus it is comforting to find

Theorem 3. *For $\alpha = 0$ and $\Sigma \cong S^1 \times \mathbb{R}$, the Hamiltonian structure of S_{DSM}, $D < E$ maximally isotropic, may be identified with the one found in Theorem 2 above.*

Proof. For a first orientation we check the statement for $D = TM$ (and $H = 0$): Then $S_{\text{DSM}} \equiv 0$. The vanishing action S (depending on whatsoever fields) is obviously equivalent (as a Hamiltonian system) to an action $\bar{S}[X, p, \lambda] = \oint (p\dot{X} - \lambda p)$ (multiplication of the integrand by $\mathrm{d}\sigma \wedge \mathrm{d}\tau$ here and below is understood). Since this example of D corresponds to $\mathcal{O} = -1$, this latter formulation obviously agrees with what one finds in Theorem 2 for this particular case. This case already illustrates that a transition from $S \equiv 0$ to $\bar{S}$ depending on additional fields as written above is an important step in establishing the equivalence.

We now turn to the general case, putting H to zero in a first step as in the proof of Theorem 2. Thus we need to analyse $S_0[X, \mathcal{A}] = \int_\Sigma A_i \wedge \mathrm{d}X^i - \frac{1}{2}A_i \wedge V^i$, with $\mathcal{A} = A \oplus V$ taking values in D. Using the unconstrained field $\mathtt{a} \equiv \mathtt{a}_0 \mathrm{d}\tau + \mathtt{a}_1 \mathrm{d}\sigma$ of (53), one finds

$$\bar{S}_0[X, p, \mathtt{a}_0, \mathtt{a}_1, \lambda] = \oint \Big(p\dot{X} - \mathtt{g}(\lambda, p - (1+\mathcal{O})\mathtt{a}_1) - \mathtt{g}(\mathtt{a}_0, (1+\mathcal{O}^{-1})\partial X + (\mathcal{O} - \mathcal{O}^{-1})\mathtt{a}_1)\Big) . \qquad (74)$$

Here the third term just collects all terms proportional to $\mathtt{a}_0$ of S_0, as one may show by a straightforward calculation (using the orthogonality of $\mathcal{O}$). The first two terms result from $\oint \mathtt{g}((1+\mathcal{O})\mathtt{a}_1, \dot{X})$, the only appearance of τ-derivatives in S_0. The transition from S_0 to $\bar{S}_0$ is the obvious generalization of the analogous step from $S \equiv 0$ to $\bar{S}$ mentioned above and explains the appearance of the new fields p and λ; eliminating these fields one obviously gets back S_0. Next we shift λ according to $\lambda = \bar{\lambda} + (1-\mathcal{O})\mathtt{a}_0$. This yields

$$\bar{S}_0[X, p, \mathtt{a}_0, \mathtt{a}_1, \bar{\lambda}] = \oint \big(p\dot{X} - \mathtt{g}(\bar{\lambda}, p - (1+\mathcal{O})\mathtt{a}_1) \\ -\mathtt{g}(\mathtt{a}_0, (1+\mathcal{O}^{-1})\partial X + (1-\mathcal{O}^{-1})p)\big) . \quad (75)$$

Note that the last term is already of the form $-\mathtt{g}(\mathtt{a}_0, \mathcal{O}^{-1}J)$, with J given by Eq. (68). We now will argue that the second term, containing the fields $\mathtt{a}_1$ and $\bar{\lambda}$, can be dropped. For the case that $1+\mathcal{O}$ is invertible, this is immediate since then the quadratic form for $\bar{\lambda}$ and $\mathtt{a}_1$ in (75) is nondegenerate—$\bar{\lambda}$ and $\mathtt{a}_1$ become completely determined by the remaining fields, without constraining them (cf. also [9]). Otherwise the variation w.r.t. $\mathtt{a}_1$ constrains the momentum p, but this constraint is fulfilled automatically by $J = 0$, resulting from the variation w.r.t. $\mathtt{a}_0$. To turn the last argument into an honest off-shell argument, we perform another shift of variables: With $\mathtt{a}_0 := \mathcal{O}^{-1}\widetilde{\lambda} - \frac{1}{2}\lambda$ and $\mathtt{a}_1 := \bar{\mathtt{a}}_1 + \frac{1}{2}\mathcal{O}^{-1}(p - \partial X)$, Eq. (75) becomes

$$\bar{S}_0[X, p, \widetilde{\lambda}, \bar{\mathtt{a}}_1, \bar{\lambda}] = \oint \big(p\dot{X} - \mathtt{g}(\widetilde{\lambda}, J) + \mathtt{g}(\bar{\lambda}, (1+\mathcal{O})\bar{\mathtt{a}}_1)\big) . \quad (76)$$

Now it is completely obvious that the last term, the only appearance of $\bar{\lambda}$ and $\bar{\mathtt{a}}_1$, can be dropped. This concludes our proof, since the remaining integrand agrees with $\mathcal{L}_{\mathrm{Ham}}$ in (73). □

As a rather immediate but important consequence the above results imply

Theorem 4. *The reduced phase space of S_{DSM} (for $\Sigma = S^1 \times \mathbb{R}$ and D any maximally isotropic $D < E$) does not depend on the choice of $\alpha \in \mathbb{R}$, the metrics h and $\mathtt{g}$ on Σ and M, respectively, or the splitting $\sigma : TM \to E$. It only depends on the subbundle D.*

Proof. Independence of α follows from the above two theorems. Independence of h is obvious and likewise the one of $\mathtt{g}$ when the constraints are written as $J_{\omega,v}(\sigma) = 0$, $\forall \omega \oplus v \in D$, cf. Theorem 2 above. Independence on the choice of the splitting is not so obvious at first sight: the symplectic form (67) depends on H (and not only on its cohomology class), and so implicitly on the splitting, cf. Eq. (28); but likewise do the constraint functions $J_{\omega,v}(\sigma) = \omega_i\,\partial X^i + v^i p_i$, since the presentation of an element of $D < E$ as $\omega \oplus v \in T^*M \oplus TM$ assumes an embedding of TM into E (while $T^*M < E$ can be identified canonically as the kernel of ρ^*, cf. Eq. (27) as well as our discussion in Sect. 4 above). The coordinate transformation on phase space $p_i \mapsto p_i + B_{ij}\,\partial X^j$ establishes the isomorphism between the two Hamiltonian structures corresponding to different splittings (cf. also [2]). Alternatively we may infer splitting independence also from Propositions 4 and 5 above. □

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