Higher Principal Bundles with Connections and Their Applications

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[1]Latest version: `https://christiansaemann.de/teaching/2026-goettingen/`

1. Introduction

These are the lecture notes to a series of four 1-hour lectures on the definition of connections on higher principal bundles and their applications in physics. The presentation is quite condensed, but I hope that it provides a good starting point for further exploration.

The included references are not meant to attribute credit and they will not be complete; please look at the listed papers for accurate and complete references. A good summary account is also found in [BJFJ+25].

1.1. Disclaimer

My background is in mathematical physics; I'm not a higher category theorist! I simply need some advanced mathematical constructions which are not readily found in the literature, and I use (Higher) Category Theory as the heuristics that allow me to identify the correct definitions and statements. For me, category theory is, as John Baez said, "the subject where definitions can be left as exercises." My conclusion from defining higher connections is that you often have to be careful how you formulate things. Certain definitions go through as naively expected, others, such as the definition of non-flat connections on higher principal bundles, require additional input.

1.2. Outline

Roughly, the lectures will go over the following topics:

- Motivation
- Higher groups
- Higher Lie algebras
- Local and global flat connection forms
- Non-flat connections and adjustments
- Examples and applications

2. Motivation

2.1. Mathematical perspective

Differential geometry has been an enormously useful bridge between mathematics and theoretical high-energy physics.

It is therefore natural to expect that higher differential geometry will have a similarly useful role, allowing for the translation of interesting observations and connections from mathematics to physics and vice versa.

2.2. Physical perspective

Let us be a bit more detailed here. We start with the most common form of a gauge theory, electromagnetism over space-time $\mathbb{R}^4$. This governs the dynamics of light, general electromagnetic waves and electrically charged particles.

In quantum mechanics, we describe an electron by a "wave function" ψ, that is a section of the trivial complex line bundle $\mathbb{R}^4 \times \mathbb{C} \to \mathbb{R}^4$. The probability to find the electron within a particular space-time volume is then simply the integral over the probability density $|\psi(x)|^2$.

Simplifying things only slightly, there is the following relation between the wave function at different points x, y:

$$\psi(y) = \underbrace{\int_{x \to y} \mathcal{D}\gamma}_{\substack{\text{sum over paths} \\ \gamma \text{ from } x \text{ to } y}} \quad \underbrace{\exp(S_{\text{free}}(\gamma))}_{\text{"free action"}} \quad \underbrace{\chi(\gamma)}_{\text{EM contribution}} \quad \psi(x) \ . \tag{2.1}$$

Here $\int_{x \to y} \mathcal{D}\gamma$ is the sum or integral over all paths γ from x to y (we are not interested in the precise definition), $\exp(S_{\text{free}}(\gamma))$ is the "free action", the part that ensures that the electron is following the correct free motion, and $\chi(\gamma)$ is the contribution from external electromagnetic fields.

For this description to make sense, we have to be able to split paths by inserting points z and summing over all possible z. As a consequence, $\chi(\gamma)$ has to be a functor, usually called the parallel transport functor, mapping the path groupoid $\mathscr{P}(\mathbb{R}^4)$, which has objects the points in $\mathbb{R}^4$ and morphisms the paths (not parametrised paths, but actual embedded paths), to the delooping $\mathsf{BU}(1)$ of the group $\mathsf{U}(1)$, which is the one-object groupoid with $\mathsf{U}(1)$ being the morphisms[2]

$$\mathscr{P}\mathbb{R}^4 \quad := \quad \begin{tikzcd} \text{paths} \arrow[r, "\chi"] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{U}(1) \arrow[d, shift left] \arrow[d, shift right] \\ \mathbb{R}^4 \arrow[r] & * \end{tikzcd} \quad = \quad \mathsf{BU}(1) \ . \tag{2.2}$$

This functor can now be written as

$$\chi(\gamma) = \exp\left(\int_\gamma A\right) \tag{2.3}$$

for some $A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{u}(1)$, which we call the local connection (1-)form. The curvature of this connection is $F = \mathrm{d}A$, which contains the components of the electric and magnetic field strengths.

Recall that Maxwell's equations read as

$$\underbrace{\mathrm{d}F = 0}_{\text{kinematics}} \quad \text{and} \quad \underbrace{\mathrm{d}^\dagger F = J}_{\text{dynamics}} \tag{2.4}$$

with the first one simply ensuring that F locally originates from a connection by the Poincaré lemma, and is hence part of the kinematics. The second equation describes the sourcing of electromagnetic fields by charged particles and currents, and describes the dynamics. In these lectures, we will be exclusively concerned with kinematical data of gauge theories, so we will ignore dynamics.

[2]see also Victoriya's lectures

More complete physical descriptions of nature, such as the Standard Model of Particle Physics, are a bit more involved. Restricting ourselves to ordinary space-time $\mathbb{R}^4$, particles are described by sections of a trivial bundle $\mathbb{R}^4 \times V \to \mathbb{R}^d$, where V carries a representation of the structure group or "gauge group" G.

The parallel transport functor is now the obvious generalisation

$$\chi : \mathscr{P}\mathbb{R}^4 \to \mathsf{BG} = (\mathsf{G} \rightrightarrows *) \ , \tag{2.5}$$

but the local connection 1-form $A \in \Omega^1(M) \otimes \mathfrak{g}$ for $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$ the Lie algebra of G yields χ via the "path-ordered exponential"[3]

$$\chi(\gamma) = P \exp \left(\int_\gamma A \right) \ . \tag{2.6}$$

Also the expression for the field strength 2-form is more involved:

$$F = \mathrm{d}A + \tfrac{1}{2}[A, A] \ . \tag{2.7}$$

Over general manifolds M, we first define a (sufficiently fine) surjective submersion $\sigma : Y \to M$. One may think of $Y = \sqcup_i U_i$, where U_i are the local patches of the manifold M, as an example. We can then consider a local parallel transport $\chi : \mathscr{P}Y \to \mathsf{BG}$ as above, and glue the local parallel transport over $Y^{[2]} = Y \times_M Y$ by a functor from the Čech groupoid $\check{\mathscr{C}}(\sigma)$ to BG:

$$\check{\mathscr{C}}(\sigma) \quad := \quad \begin{array}{ccc} Y^{[2]} & \xrightarrow{g} & \mathsf{G} \\ \downdownarrows & & \downdownarrows \\ Y & \longrightarrow & * \end{array} \quad = \quad \mathsf{BG} \tag{2.8}$$

according to

$$\chi_{U_1}(\gamma) = (g(y_1, y_2))^{-1} \chi_{U_2}(\gamma) g(y_1, y_2) \ , \tag{2.9}$$

where γ is a path from y_2 to y_1, completely contained in the intersection $U_1 \cap U_2 \subset Y \times_M Y$ of the sets $U_{1,2} \subset Y$.

We note that the functor g defines nothing but the Čech cocycle of the transition functions of a principal G-bundle. Recall that such a principal bundle is a surjective submersion $\pi : P \to M$ with a smooth, free, and transitive right-action $P \times \mathsf{G} \to P$ such that $M = P/\mathsf{G}$ and with P locally diffeomorphic to $U \times \mathsf{G}$, $U \subset M$.

As a side remark, we note that the Lie groupoid $\check{\mathscr{C}}(\sigma)$ is weakly equivalent (in the bicategory of groupoids, bibundles, and bibundle isomorphisms) to the manifold M, trivially regarded as a Lie groupoid $M \rightrightarrows M$, so g is really a bibundle or weak morphism from M to BG. This matches the interpretation of a principal bundle as a map from M to the classifying space BG of principal G-bundles.

We presented the above "glued local" notion of parallel transport, since this is how we will want to look at higher principal bundles with connections later (and it is also the way that physicists think about principal bundles, since they want to impose differential

[3] This is the solution to the differential equation $\chi(\gamma_x)\mathrm{d}(\chi(\gamma_x))^{-1} = \chi(\gamma_x)A(x)(\chi(\gamma_x))^{-1}$, where γ_x is a path that starts at some chosen base point p and ends at x.

equations on the local gauge potential forms). A global version of the parallel transport is the global parallel transport functor from $\mathscr{P}M$ to the Atiyah groupoid:

$$\chi : \mathscr{P}M \to (P \times_{\mathsf{G}} P \rightrightarrows M) \ . \tag{2.10}$$

We also note that a connection can be seen equivalently as a smooth G-equivariant decomposition of the tangent bundle TP into a horizontal subbundle H, which complements the vertical bundle $V = \ker(\mathrm{d}\pi)$ at every point:

$$TP_p = H_p \oplus V_p \ . \tag{2.11}$$

2.3. But why higher bundles?

In string/M-theory, we have higher-dimensional objects such as strings, D-branes, or M-branes, whose movement through spacetime sweeps out surfaces and higher dimensional volumes. The field content of these theories also contains the corresponding higher-degree local connection forms. We therefore need a higher-dimensional parallel transport.

Clearly, a higher-dimensional parallel transport needs to be reparametrisation invariant. That is, it does not matter in which order we parallel transport e.g. two halves of a string:

$$\begin{tikzcd}[column sep=large]
\bullet & {} \arrow[l, bend right=60] \arrow[l] \arrow[l, bend left=60] \arrow[l, phantom, shift right=3ex, "\Downarrow g_1"] \arrow[l, phantom, shift left=3ex, "\Downarrow g_1'"] & \bullet \arrow[l, bend right=60] \arrow[l] \arrow[l, bend left=60] \arrow[l, phantom, shift right=3ex, "\Downarrow g_2"] \arrow[l, phantom, shift left=3ex, "\Downarrow g_2'"]
\end{tikzcd} \tag{2.12}$$

We can either transport both halves to the middle string and then further or first completely transport the first half and then completely transport the second half. Assuming that the parallel transport is described by group elements, we obtain the equation

$$(g_1' \otimes g_2') \circ (g_1 \otimes g_2) = (g_1' \circ g_1) \otimes (g_2' \circ g_2) \ . \tag{2.13}$$

This forces the g_i to be abelian by the so-called "Eckmann/Hilton-argument".

Exercise 2.1. Derive this argument by first showing that both products are the same product and then showing that this product is abelian.

This, however, is too restrictive for our purposes, particularly in physics, where interactions often result from non-abelian contributions.

The obvious way out of this problem is to simply work with higher categories, and to interpret the string as 1-morphisms between points, and the parallel transport as 2-morphisms between these strings.

Exercise 2.2. Why does the EH-argument not apply in the case of 2-categories?

So the roadmap is clear: we need to develop categorified principal bundles with categorified structure groups and categorified connections, which will be differential forms taking values in categorified Lie algebras.

We note that there is[4] a misconception, particularly among theoretical physicists that higher principal bundles with connections either do not exist or do not make sense, or

[4] Hopefully, "has been" is more accurate by now.

are just limited to the abelian case. This is definitely not true! (But given the subtleties in the correct definition of higher non-abelian connections, it's understandable how this misconception arose.)

3. Higher principal bundles, topologically

3.1. Higher groups

When categorifying a mathematical notion, there are usually several different perspectives, which are all related. For higher groups, we can use, for example, the following:

- Group objects in higher categories: the straightforward, vertical categorification of the notion of a group
- Higher groupoids with a single object: you may have a more general notion of higher groupoids. If these have a single object, you have a higher group.
- Simplicial groups: that is, a simplicial object in the category of groups. This yields strict models that allow for explicit computations.
- Hypercrossed modules: these are obtained from simplicial groups by taking the Moore complex, which removes redundancy in the simplicial description:

$$N_\bullet(\mathsf{G}) = \bigcap_{i=1}^{n} \ker(\mathsf{d}_i^n)~, \quad \partial_\bullet = \mathsf{d}_0^\bullet~. \tag{3.1}$$

  This Moore complex is then naturally endowed with further structure, such as e.g. an action of $N_0(\mathsf{G})$ on all elements.

For the description of higher groups, we will limit ourselves to the case higher group=2-group, which already exposes all the key features.

Definition 3.1. A (weak) 2-group is a group object in the 2-category of groupoids.

Equivalently, it is a weak monoidal category where all objects are weakly invertible and all morphisms are invertible.

Such general 2-groups can be very hard to work with explicitly, see e.g. the involved discussion of the string 2-group in [Sch11]. We will therefore work with strict 2-groups.

Definition 3.2. A strict 2-group is a simplicial group with a Moore complex of length 2.

Equivalently, it is the nerve of a strict monoidal category with strict inverses of objects and morphisms.

The Moore complex of such a 2-group can be extended to a crossed module of groups:

Definition 3.3. A crossed module of groups $(\mathsf{H} \xrightarrow{\partial} \mathsf{G}, \rhd)$ consists of a pair of groups G, H, a group morphism $\partial : \mathsf{H} \to \mathsf{G}$, and an action $\rhd$ of G on H by automorphisms such that

$$\partial(g \rhd h_1) \;=\; g\partial(h_1)g^{-1} \quad \text{and} \quad \partial(h_1) \rhd h_2 \;=\; h_1h_2h_1^{-1} \tag{3.2}$$

for all $g \in \mathsf{G}$ and $h_{1,2} \in \mathsf{H}$.

This equivalently describes strict 2-groups:

Proposition 3.4. *[BS76] The 2-category of strict 2-groups and the 2-category of crossed modules are equivalent.*

Let us give a list of examples of crossed modules $(\mathsf{H} \xrightarrow{\partial} \mathsf{G}, \triangleright)$:

- We can regard an ordinary group K trivially as a crossed module with $\mathsf{H} = *$, $\mathsf{G} = \mathsf{K}$.
- The delooping BK of an abelian group K, which we previously interpreted as the one-object groupoid $(\mathsf{K} \rightrightarrows *)$ is a crossed module with $\mathsf{H} = \mathsf{K}$ and $\mathsf{G} = *$. We need to restrict to abelian groups, since the axioms (3.2) imply that the kernel of ∂ is abelian.
- The automorphism 2-group of a group K was the first one for which higher principal bundles for strict 2-groups were written down in [BM05, ACJ05]. Here, $\mathsf{H} = \mathsf{K}$ and $\mathsf{G} = \mathsf{Aut}(\mathsf{K})$.

3.2. Morphisms of higher groups

A common feature of categorified objects is that their morphisms are significantly richer than their uncategorified counterparts. Usually, it is easy to define strict morphisms, which preserve structures on the nose. More often, and in particular in the definition of higher principal bundles, however, we will need weak functors, in which structures are preserved up to higher corrections.

In the case of crossed modules, the naturally expected spans of strict morphisms can be reduced to the following notion of morphism:

Definition 3.5. [Noo, AN09, AN10] Given two crossed modules of groups $(\mathsf{H}_{1,2} \xrightarrow{\partial_{1,2}} \mathsf{G}_{1,2}, \triangleright_{1,2})$, a butterfly is a commutative diagram of groups

$$\begin{tikzcd}
\mathsf{H}_1 \arrow[dd, "\partial_1"'] \arrow[dr, "\lambda_1"] & & \mathsf{H}_2 \arrow[dd, "\partial_2"] \arrow[dl, "\lambda_2"'] \\
& \mathsf{E} \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma_2"] & \\
\mathsf{G}_1 & & \mathsf{G}_2
\end{tikzcd} \tag{3.3a}$$

where E is a group, the NE–SW diagonal is a short exact sequence (i.e. a group extension), the NW-SE diagonal is a complex, and the following compatibility relation holds

$$\lambda_{1,2}(\gamma_{1,2}(e) \triangleright_{1,2} h_{1,2}) \;=\; e\lambda_{1,2}(h_{1,2})e^{-1} \;. \tag{3.3b}$$

A flippable butterfly is a butterfly where both diagonals are short exact sequences.

Flippable butterflies describe weak equivalences between crossed modules of groups.

Exercise 3.1. Show that $\ker(\mathsf{t})$ and $\operatorname{coker}(\mathsf{t})$ are invariant under weak equivalences.

The restriction of the above definition of 2-groups and their morphisms to Lie 2-groups should be evident; we will ignore the subtlety in defining a nice category of smooth spaces; generally, diffeological spaces should be sufficient.

As an example of a weak equivalence, let us consider a non-trivial way of regarding an ordinary Lie group as a crossed module of Lie groups. Consider the crossed module

$$\mathcal{L}\mathsf{G} = (L_0\mathsf{G} \xrightarrow{e} P_0\mathsf{G}) \ , \tag{3.4}$$

where $P_0\mathsf{G}$ and $L_0\mathsf{G}$ denote the based, parametrised path and loop spaces[5] of G. The map e is the embedding, the group product is the pointwise multiplication of the parametrised paths, and the action is by pointwise adjoint action.

There is now the following flippable butterfly:

$$\begin{tikzcd} L_0\mathsf{G} \arrow[dd, "\partial"'] \arrow[dr, hook, "e"] & & * \arrow[dl, "\mathbb{1}"'] \arrow[dd, "\mathbb{1}"] \\ & P_0\mathsf{G} \arrow[dl, "\mathsf{id}"'] \arrow[dr, "\flat"] & \\ P_0\mathsf{G} & & \mathsf{G} \end{tikzcd} \tag{3.5}$$

which describes an equivalence between the crossed modules $\mathcal{L}\mathsf{G}$ and $(* \to \mathsf{G})$.

Exercise 3.2. Show the compatibility relation for the butterfly.

The above example will serve as an interesting test case for our notions of higher principal bundle with connection. In particular, we expect that the data for principal G-bundles is equivalent to that of principal $\mathcal{L}\mathsf{G}$-bundles.

3.3. Čech cohomology for higher principal bundles

Abstractly, the definition of a higher principal bundle is clear: this is a higher space with a higher principal action of a higher group, which is locally trivial.

Here, we are interested in the description in terms of Čech cohomology or transition functions, and we can use the obvious extension of what we encountered in section 2.2. For this, we need the fact that a crossed module of Lie groups is equivalently a monoidal category of the form $\mathsf{H} \rtimes \mathsf{G} \rightrightarrows \mathsf{G}$, which allows us to define a delooping 2-groupoid $\mathsf{B}\mathcal{G}$. The full translation is found in [BL04], a translation in the convention that we use here is in [JSW15].

Definition 3.6. Consider a crossed module of Lie groups $\mathcal{G} = (\mathsf{H} \xrightarrow{\partial} \mathsf{G}, \triangleright)$ and a surjective submersion of manifolds $\sigma : Y \to M$. A <u>Čech cocycle</u> for a principal $\mathcal{G}$-bundle over M and subordinate to σ is a smooth weak 2-functor

$$\check{\mathscr{C}}(\sigma) \to \mathsf{B}\mathcal{G} \ . \tag{3.6}$$

[5] We ignore all technical details. These are found, e.g., in [SW11].

An isomorphism or Čech 1-coboundary between principal $\mathcal{G}$-bundles is a natural 2-isomorphism between the weak functors. A 2-isomorphism or a Čech 2-coboundary is a 2-modification between the natural 2-isomorphisms.

Concretely, one can show that a principal $\mathcal{G}$-bundle is equivalently a pair of maps[6] $g \in C^\infty(Y^{[2]}, \mathsf{G})$ and $h \in C^\infty(Y^{[3]}, \mathsf{H})$ such that

$$\begin{aligned} g_{ac} &= \partial(h_{abc}) g_{ab} g_{bc} \ , \\ h_{acd} h_{abc} &= h_{abd} (g_{ab} \triangleright h_{bcd}) \ , \end{aligned} \tag{3.7}$$

where a, b, c, d denote points in Y or the various fiber products. The first equation is the usual equation for a principal G-bundle, deformed by a 2-morphism $\partial(h_{abc})$. This is again the common feature of categorification: equations do not hold on the nose, but up to higher morphisms. The second equation is a consistency equation that is obtained by gluing four versions of the first equation across a tetrahedron with corners a, b, c, and d, and assuming that ∂ is injective.

Exercise 3.3. Derive the above Čech cocycle relations from the corresponding weak 2-functor. What are the corresponding Čech 1- and 2-coboundaries describing bundle 1- and 2-isomorphisms?

As examples, let us consider two extreme cases for now.

- A $\mathcal{G}$-principal bundle with $\mathsf{H} = *$ is simply a principal G-bundle.
- For $\mathsf{G} = *$, we have an abelian gerbe[7] with "band" H, as defined e.g. in [Hit99].

If you are interested in using simplicial groups directly, you should use the notion of twisted Cartesian product, as found in [May93]. For a simplicial group with Moore complex of length 2, this reproduces the above cocycles.

4. Higher Lie algebras

4.1. Differentiating higher Lie groups

For the description of local connection forms, we also require a good notion of higher Lie algebras. In fact, the description we develop will almost directly lead to a notion of local connection form for higher principal bundles.

First, we note that applying the tangent functor to a crossed module of Lie groups $\mathcal{G} = (\mathsf{H} \xrightarrow{\partial} \mathsf{G}, \triangleright)$, we obtain the notion of a crossed module of Lie algebras $\mathsf{Lie}(\mathcal{G}) = (\mathfrak{h} \xrightarrow{\partial} \mathfrak{g}, \triangleright)$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H.

More generally, we can differentiate simplicial Lie groups to simplicial Lie algebras, which is the same as applying the tangent functor to the corresponding hypercrossed module.

Strict higher Lie algebras are nothing but differential graded Lie algebras.

[6] Here, $Y^{[n]}$ denotes the fibre product $Y \times_M Y \times_M \ldots \times_M Y$ with n factors.

[7] It's good to be also aware of the more geometric description in terms of bundle gerbes [Mur07].

Exercise 4.1. Show that a crossed module of Lie algebras $(\mathfrak{h} \xrightarrow{\partial} \mathfrak{g}, \triangleright)$ is equivalently a differential graded Lie algebra (dgLa) $\mathfrak{G} = (\mathfrak{G}_{-1} \xrightarrow{\mu_1} \mathfrak{G}_0, \mu_2)$ with $\mathfrak{G}_{-1} = \mathfrak{h}$ and $\mathfrak{G}_0 = \mathfrak{g}$.

More generally, higher Lie groups differentiate to L_∞-algebras [Zwi93, LS93, LM95]. A concrete algorithm for this differentiation procedure was given by Ševera in [Šev06], which produces the Chevalley–Eilenberg algebra of the expected L_∞-algebra. An extended review of the following in great detail and conventions that match ours here is found in [JRSW19].

4.2. L_∞-algebras from Chevalley–Eilenberg algebras

Recall that the Chevalley–Eilenberg algebra of a Lie algebra $\mathfrak{g}$ is the semifree[8] differential graded commutative algebra (dgca)[9]

$$C^\infty(\mathfrak{g}[1]) = \odot^\bullet \mathfrak{g}[1]^* \ , \tag{4.1}$$

whose differential encodes the differential and the brackets.

Concretely, $\mathsf{CE}(\mathfrak{g})$ is generated by $\mathfrak{g}[1]^*$, and the differential d_{CE} is defined by its action on $\mathfrak{g}[1]^*$. For simplicity[10], we will choose a basis of generators

$$t^\alpha \ , \quad \alpha = 1, \ldots, \dim \mathfrak{g} \ , \tag{4.2}$$

which are all of degree 1, $|t^\alpha| = 1$. The most general differential of degree 1 is then given by

$$\mathsf{d}_{\mathsf{CE}} t^\alpha = -\tfrac{1}{2} \left(\sum_{\beta,\gamma} \right) f^\alpha_{\beta\gamma} t^\beta t^\gamma \ , \tag{4.3}$$

where $f^\alpha_{\beta\gamma}$ are some constants in the ground field over which we are working. These constants define the Lie bracket on $\mathfrak{g}$ in the basis e_α, dual to t^α (and grade-shifted):

$$[e_\alpha, e_\beta] = \left(\sum_\gamma \right) f^\gamma_{\alpha\beta} e_\gamma \ , \tag{4.4}$$

where it is customary in physics to drop the explicit summation symbol. This is called "Einstein summation convention", and comes from the observation that almost always, an index appearing twice is summed over as a consequence of covariance of expressions. The condition $\mathsf{d}_{\mathsf{CE}}^2 = 0$ is then equivalent to the Jacobi identity for the Lie bracket.

An obvious generalisation is now to replace $\mathfrak{g}$ by an arbitrary $\mathbb{Z}$-graded vector space $\mathfrak{L}$. The most general differential d_{CE} then encodes operations, also called "higher products",

$$\mu_n : \mathfrak{L}^{\wedge n} \to \mathfrak{L} \tag{4.5}$$

[8] i.e. free in the product, but the differential satisfies relations

[9] Here, we use the convention that for a graded vector space $V = \oplus_k V_k$, the n-shifted vector space is $V[n] := \oplus_k V[n]_k$ with $V[n]_k = V_{k+n}$.

[10] My apologies for this; there is certainly a coordinate-independent description, which, however, leads to a less intuitive description. Also, this is also close to how physicists would usually work, and to talk to them, it is always helpful to know a bit more about their language.

of degree $2-n$, and the condition $\mathsf{d}_{\mathsf{CE}}^2 = 0$ yields the homotopy Jacobi identities.[11]

As a concrete example, let us consider a 2-term L_∞-algebra:

$$\mathfrak{L} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_0\ , \quad \mathsf{CE}(\mathfrak{L}) = \odot^\bullet(\mathfrak{L}_{-1} \oplus \mathfrak{L}_0)[1]^*\ . \tag{4.6}$$

Here, we have generators t^α and s^a of degrees $|t^\alpha| = 1$ and $|s^a| = 2$, and the most general differential reads as

$$\begin{aligned} \mathsf{d}_{\mathsf{CE}} t^\alpha &= \pm f_a^\alpha s^a \pm \tfrac{1}{2} f_{\beta\gamma}^\alpha t^\beta t^\gamma \\ \mathsf{d}_{\mathsf{CE}} s^a &= \pm f_{\alpha b}^a t^\alpha s^b + f_{\alpha\beta\gamma}^a t^\alpha t^\beta t^\gamma \end{aligned} \tag{4.7}$$

for constants $f_{\dots}^{\dots}$ taking values in the ground field. Dually, the constants define higher products

$$\begin{aligned} f_a^\alpha \to \quad & \mu_1 : & \mathfrak{L}_{-1} \to \mathfrak{L}_0\ , \\ f_{\alpha\beta}^\gamma \to \quad & \mu_2 : & \mathfrak{L}_0 \times \mathfrak{L}_0 \to \mathfrak{L}_0\ , \\ f_{\alpha a}^b \to \quad & \mu_2 : & \mathfrak{L}_0 \times \mathfrak{L}_{-1} \to \mathfrak{L}_{-1}\ , \\ f_{\alpha\beta\gamma}^a \to \quad & \mu_3 : & \mathfrak{L}_0 \times \mathfrak{L}_0 \times \mathfrak{L}_0 \to \mathfrak{L}_{-1}\ , \end{aligned} \tag{4.8}$$

From the condition $\mathsf{d}_{\mathsf{CE}}^2 = 0$, we generally get the following:

- μ_1 is a (co-)differential, so we have the underlying (co-)chain complex

$$\dots \xrightarrow{\mu_1} \mathfrak{L}_{-1} \xrightarrow{\mu_1} \mathfrak{L}_0 \xrightarrow{\mu_1} \mathfrak{L}_1 \xrightarrow{\mu_1} \dots \tag{4.9}$$

- The differential μ_1 is a derivation of the binary product μ_2 (in fact, μ_1 is a derivation in the obvious sense of all the higher products μ_n).
- The Jacobi identity is violated up to a homotopy $\mu_1 \circ \mu_3 + \mu_3 \circ \mu_1$.

Exercise 4.2. Show the above.

It sometimes seems difficult to get the signs right ("The hardest problem in higher gauge theory is not to identify the various moduli spaces, but to get all minus signs right...") but they usually follow directly from respecting the Koszul sign rule: whenever you move something odd past something odd, you pick up a minus sign.

Interesting classes of L_∞-algebras are the following:

- Lie algebras and all dgLas are strict L_∞-algebras, i.e. $\mu_n = 0$ for $n \geq 3$.
- Differential complexes are "abelian" L_∞-algebras with $\mu_n = 0$ for $n \geq 2$.
- Minimal L_∞-algebras are those with $\mu_1 = 0$.
- "Semistrict" Lie 2-algebras: L_∞-algebras "concentrated" (i.e. have non-trivial elements) in degrees -1 and 0.
- "Semistrict" Lie n-algebras: L_∞-algebras concentrated in degrees $-n+1, \dots, 0$.

The description of L_∞-algebras in terms of Chevalley–Eilenberg algebras immediately yields the notion of weak morphisms (cf. e.g. [JRSW19]):

[11]There is a much longer story behind this, where for each operad ($\to$ Claudia's lectures) one can construct the corresponding homotopy algebra using the Koszul dual. Here, the Lie-operad is Koszul dual to the Com-operad, and this is why the dual description as a Chevalley–Eilenberg algebra is a semifree dgca.

Definition 4.1. A weak morphism of L_∞-algebras $\phi : \mathfrak{L} \to \tilde{\mathfrak{L}}$ is the dual of a morphism of cdgas: $\phi^* : \mathsf{CE}(\tilde{\mathfrak{L}}) \to \mathsf{CE}(\mathfrak{L})$.

Translating this to the bracket picture, we obtain a family of maps

$$\phi_k : \mathfrak{L}^{\wedge k} \to \tilde{\mathfrak{L}} \quad \text{for} \quad 1 \leq k \in \mathbb{N} \ , \tag{4.10}$$

such that ϕ_1 is a chain map, and the other ϕ_k describe the failure of ϕ_1 to preserve brackets μ_n.

Exercise 4.3. Derive the relations in lowest degree, e.g. for a 2-term L_∞-algebra.

Recall that quasi-isomorphisms of chain complexes are special chain maps that induce an isomorphism between the cohomologies. We extend this to L_∞-algebras:

Definition 4.2. A morphism of L_∞-algebras $\phi : \mathfrak{L} \to \tilde{\mathfrak{L}}$ is called a quasi-isomorphism, if the contained chain map ϕ_1 induces an isomorphism between the cohomologies of the L_∞-algebras.

Quasi-isomorphisms describe weak equivalences of L_∞-algebras. In particular, applying the differential to the butterflies that describe weak morphisms between crossed modules of Lie groups yield morphisms of crossed modules of Lie algebras which are very mildly more general than the quasi-isomorphisms of the corresponding strict L_∞-algebras.

There are now two important structural theorems for L_∞-algebras: the strictification theorem and the minimal model theorem:

Proposition 4.3. *(e.g. [BM07]) Every L_∞-algebra is quasi-isomorphic to a strict L_∞-algebra (i.e. $\mu_n = 0$ for $n \geq 3$).*

Proposition 4.4. *(e.g. [Kad82, Kaj07]) Every L_∞-algebra is quasi-isomorphic to a minimal one (i.e. $\mu_1 = 0$).*

Let us discuss the archetypal example of the string Lie 2-algebra:

- There is a minimal 2-term L_∞-algebra modelling the string Lie 2-algebra[12]:

  $$\mathfrak{L}^\circ = \big(\ \mathbb{R} \xrightarrow{0} \mathfrak{spin}(n)\ \big) \ , \tag{4.11}$$

  and the non-trivial higher products are

  $$\mu_2(x_1, x_2) = [x_1, x_2] \ , \quad \mu_3(x_1, x_2, x_3) = \tfrac{1}{3}(x_1, [x_2, x_3]) \tag{4.12}$$

  for $x_i \in \mathfrak{spin}(n)$, where $(-,-)$ denotes the Killing form.

[12]This example readily extends to any quadratic Lie algebra.

- There is a quasi-isomorphic strict 2-term L_∞-algebra

$$\mathfrak{L}^{\rm str} = \Big(\mathbb{R} \oplus L_0\mathfrak{spin}(n) \xrightarrow{\mu_1} P_0\mathfrak{spin}(n) \Big) \tag{4.13}$$

with non-trivial higher products given by

$$\begin{aligned} \mu_1(r,\lambda) &= \lambda \in P_0\mathfrak{spin}(n) \ , \\ \mu_2(\gamma_1,\gamma_2) &= [\gamma_1,\gamma_2] \ , \\ \mu_2(\gamma,(r,\lambda)) &= \Big([\gamma,\lambda], -2\int_0^1 \mathrm{d}\tau\Big(\gamma(\tau),\frac{\mathrm{d}}{\mathrm{d}\tau}\lambda(\tau)\Big)\Big) \ . \end{aligned} \tag{4.14}$$

- The chain map encoding the quasi-isomorphism between these is

$$\begin{tikzcd} L_0\mathfrak{spin}(n)\oplus\mathbb{R} \arrow[r, "\mathrm{pr}_\mathbb{R}"] \arrow[d, "\mu_1"] & \mathbb{R} \arrow[d, "0"] \\ P_0\mathfrak{spin}(n) \arrow[r, "\flat"] & \mathfrak{spin}(n) \end{tikzcd} \tag{4.15}$$

where $\flat$ denotes the boundary map mapping a path to its endpoint.

Exercise 4.4. Complete this quasi-isomorphism. Using some choice of injective, smooth map $\wp : [0,1] \to [0,1]$ with $\wp(0) = 0$ and $\wp(1) = 1$, define the inverse quasi-isomorphism.

5. Connections

5.1. Flat local connection forms

Recall that we usually define connections as splittings of the Atiyah algebroid sequence [Ati57]:

$$\begin{tikzcd} 0 \arrow[r] & \ker(\mathrm{d}\pi) \arrow[r, hook] & \mathrm{At}(P) := TP/G \arrow[r, two heads] & TM \arrow[r] \arrow[l, bend right, dashed, "A"'] & 0 \end{tikzcd} \tag{5.1}$$

The problem with this description is that this is a splitting in the category of vector bundles. If we restrict splittings to the much more natural category of Lie algebroids, we only get flat connections. Unfortunately, we will have to follow the latter approach for higher connections, as we obtain "too much data" otherwise.

We restrict ourselves to the local case (or the case of trivial P), cf. e.g. [Car50a, Car50b, KS15, SSS09].

Definition 5.1. Let $\mathfrak{L}$ be an L_∞-algebra. An L_∞-algebra-valued connection over M is a morphism of dg-manifolds

$$\mathcal{A} : (\mathrm{T}[1]M, \mathrm{d}) \to (\mathfrak{L}[1], \mathsf{d}_{\mathsf{CE}}) \quad \Leftrightarrow \quad \mathcal{A}^* : \mathsf{CE}(\mathfrak{L}) \to \Omega^\bullet(M) \ . \tag{5.2}$$

We note that for $\mathfrak{L} = \mathfrak{g}$, this is indeed a local splitting of the Atiyah-algebroid sequence as Lie algebroids.

As a quick example, consider a 2-term L_∞-algebra with generators as above. Here, we map

$$\begin{aligned} t^\alpha &\mapsto A^\alpha \in \Omega^1(M)~, \quad A \in \Omega^1(M) \otimes \mathfrak{L}_0~, \\ s^a &\mapsto B^a \in \Omega^2(M)~, \quad B \in \Omega^2(M) \otimes \mathfrak{L}_{-1}~. \end{aligned} \tag{5.3}$$

We call these the "local connection forms". The compatibility with differential amounts to

$$\begin{aligned} F &\coloneqq \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) + \mu_1(B) = 0~, \\ H &\coloneqq \mathrm{d}B + \mu_2(A,B) + \tfrac{1}{3!}\mu_3(A,A,A) = 0~. \end{aligned} \tag{5.4}$$

To obtain non-flat connections, we could have generalised to morphisms of graded manifolds, and defined the curvatures as the failure to preserve the differential.

5.2. Higher principal bundles with flat connections

For a complete description of the kinematical data of principal bundles with flat connections, we need to extend the above, local description of flat connections by gluing together the local data to global data. This is done by (higher) bundle isomorphism on overlaps $Y^{[2]}$. It turns out that a very natural way of achieving this [FJFKS24] is based on a generalisation of Ševera's differentiation procedure [Šev06].

Let $\Theta \coloneqq \mathbb{R}[-1]$, and endow this "odd line" with the natural differential

$$\mathrm{d}(a + \theta b) = b \tag{5.5}$$

for all $a, b, \in \mathbb{R}$ and θ the coordinate on $\mathbb{R}[-1]$ of degree $|\theta| = 1$. It follows that the Čech groupoid $\check{\mathscr{C}}(\sigma_\Theta)$ for the trivial submersion $\sigma_\Theta = \Theta \to *$ is a dg-Lie groupoid.

Definition 5.2. Let $\mathscr{G}$ be a higher groupoid. The corresponding inner action groupoid is defined as the inner homomorphism groupoid

$$\mathscr{A}(\mathsf{B}\mathscr{G}) \coloneqq \underline{\mathsf{dgGrpd}}(\check{\mathscr{C}}(\sigma_\Theta), \mathscr{G})~, \tag{5.6}$$

where we trivially regard $\check{\mathscr{C}}(\sigma_\Theta)$ as a higher groupoid in a suitably chosen category dgGrpd of differential groupoids that contains also $\mathscr{G}$.

Note that we can regard a higher group $\mathscr{H}$ trivially as a higher groupoid by taking its delooping: $\mathscr{G} = \mathsf{B}\mathscr{H}$. Also, any higher groupoid is trivially a differential graded groupoid, concentrated in degree 0 and with trivial differential.

As an example, consider an ordinary Lie group G. Its dg-action groupoid reads as

$$\mathscr{A}(\mathsf{BG}) = (\mathfrak{g}[1] \rtimes T[1]\mathsf{G}, \mathsf{d}_{\mathsf{CE}} + \mathrm{d}) \rightrightarrows (\mathfrak{g}[1], \mathsf{d}_{\mathsf{CE}})~, \tag{5.7}$$

where d_{CE} is the Chevalley–Eilenberg-differential on $\mathfrak{g}[1]$ and d is the de Rham differential, the natural differential on $T[1]\mathsf{G}$.

We then define:

Definition 5.3. Let $\mathscr{G}$ be a higher groupoid. Flat principal $\mathscr{G}$-bundles with connections over M and subordinate to some surjective submersion $\sigma : Y \to M$ are morphisms of (higher) dg-groupoids

$$(T[1]\check{\mathscr{C}}(\sigma), \mathrm{d}) \to \mathscr{A}(\mathsf{B}\mathscr{G}) \ . \tag{5.8}$$

Bundle isomorphisms are given by the corresponding natural isomorphisms, and higher bundle isomorphisms correspond to higher transfors.

We note that over Y, we still have the morphisms (5.2), so that this description indeed extends that of local flat connections.

As an example, let us work out the example for $\mathscr{G}$ a strict 2-group given by the crossed module of Lie groups $(\mathsf{H} \xrightarrow{\partial} \mathsf{G}, \rhd)$. The morphism of dg-groupoids contains the local connection forms

$$A \in \Omega^1(Y) \otimes \mathsf{Lie}(\mathsf{G}) \quad \text{and} \quad B \in \Omega^2(Y) \otimes \mathsf{Lie}(\mathsf{H}) \tag{5.9}$$

as well as the expected Čech data satisfying relations (3.7). The local connection forms are glued together by higher bundle isomorphisms over $Y^{[2]}$. These higher bundle isomorphisms actually form a 2-groupoid (the BRST-2-groupoid) with morphisms

$$\begin{tikzcd}[column sep=huge] (\tilde{A},\tilde{B}) & (A,B) \arrow[l, bend right=40, "{(A,B,a,\lambda)}"', ""{name=U, below}] \arrow[l, bend left=40, "{(A,B,\tilde{a},\tilde{\lambda})}", ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "{(A,B,a,\lambda,m)}"] \end{tikzcd} \tag{5.10a}$$

Here,

$$a \in \Omega^0(Y, \mathsf{G}) \ , \quad \lambda \in \Omega^1(Y) \otimes \mathsf{Lie}(\mathsf{H}) \tag{5.10b}$$

are the moduli of the 1-morphisms, which relate the local connection forms as follows:

$$\begin{aligned} \tilde{A} &= a^{-1}Aa + a^{-1}\mathrm{d}a - \partial(\lambda) \ , \\ \tilde{B} &= a^{-1} \rhd B + \mathrm{d}\lambda + \tilde{A} \rhd \lambda + \tfrac{1}{2}[\lambda, \lambda] \ . \end{aligned} \tag{5.10c}$$

Moreover,

$$m \in \Omega^0(Y, \mathsf{H}) \tag{5.10d}$$

describes the moduli of the 2-morphisms, which related 1-morphisms via

$$\begin{aligned} \tilde{a} &= \partial(m)a \ , \\ \tilde{\lambda} &= \lambda + a^{-1} \rhd (m^{-1}\mathrm{d}m + (m^{-1}A \rhd m)) \ . \end{aligned} \tag{5.10e}$$

6. General higher principal bundles with connection

The obvious question is now how to extend this to non-flat connections. In the case of ordinary principal bundles, we keep exactly the same formulas for gluing relations and bundle isomorphisms, but we simply lift the flatness condition.

This, however, creates problems in the case of higher bundles. In particular for the structure of higher bundle isomorphism (5.10), we note the following [RSW26]:

$$
\begin{tikzcd}
(\tilde{A},\tilde{B}) & & (A,B) \arrow[ll, bend right, "{(A,B,a,\lambda)}"'] \arrow[dll, bend left, "{(A,B,\tilde{a},\tilde{\lambda})}"] \arrow[dll, phantom, "\Downarrow{\scriptstyle (A,B,a,\lambda,m)}"] \\
(\tilde{\tilde{A}},\tilde{\tilde{B}}) = (\tilde{A},\tilde{B}+a^{-1}\triangleright(m^{-1}F\triangleright m)) & &
\end{tikzcd}
\tag{6.1}
$$

That is, higher gauge transformations generally break globularity, and we lose the structure of a higher action groupoid of bundle isomorphisms. This is already quite a severe breakage of expected mathematical structures. In the case of weak 2-groups, the situation is even more problematic: we would even lose consistent composition of 1-morphisms!

This is why in many discussions of higher connections, "fake flatness" is imposed. That is, all curvature forms bar the one of highest form degree vanish. Here, this would mean

$$F = \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) + \mu_1(B) \overset{!}{=} 0 \ . \tag{6.2}$$

This condition, however, will prove to be too restrictive for many purposes.

From special situations in physics, we have some guidance on what we should do to rectify the situation: we need to deform the action of the bundle isomorphisms in the curved case.

Clearly, we want to preserve the form of flat transformations, so that any deformation has to be proportional to the curvature 2-form F. The only such deformation of (5.10) is then parametrised by a map

$$\kappa : \mathsf{G}\times\mathfrak{g}\to\mathfrak{h} \tag{6.3}$$

unital in its first argument and linear in its second. Explicitly, these deformations, called <u>adjustments</u>, reads as

$$
\begin{aligned}
\tilde{A} &= a^{-1}Aa + a^{-1}\mathrm{d}a - \partial(\lambda) \ ,\\
\tilde{B} &= a^{-1}\triangleright B + \mathrm{d}\lambda + \tilde{A}\triangleright\lambda + \tfrac{1}{2}[\lambda,\lambda] - \kappa\big(a^{-1},F\big) \ ,\\
\tilde{a} &= \partial(m)a \ ,\\
\tilde{\lambda} &= \lambda + a^{-1}\triangleright(m^{-1}\mathrm{d}m + (m^{-1}A\triangleright m)) \ .
\end{aligned}
\tag{6.4}
$$

Proposition 6.1. *[RSW26] The (higher) gauge transformations glue together and have the expected globular structure if and only if*

$$
\begin{aligned}
\kappa(\partial(h),V) &= h(V\triangleright h^{-1}) \ ,\\
\kappa(g_2g_1,V) &= g_2\triangleright\kappa(g_1,V) + \kappa\big(g_2, g_1Vg_1^{-1} - \partial(\kappa(g_1,V))\big) \ .
\end{aligned}
\tag{6.5}
$$

Note that bundle isomorphisms can be interpreted as partially flat homotopies between the local maps (5.2). Therefore, a deformation of bundle isomorphism induces a deformation of curvatures, and in the case above, we have the curvature forms

$$\begin{aligned} F &\coloneqq \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) + \mu_1(B)\ , \\ H &\coloneqq \mathrm{d}B + \mu_2(A,B) + \kappa(A,F)\ . \end{aligned} \tag{6.6}$$

For the full principal bundles, we also have to adjust the gluing conditions for the local connection 2-form B over $Y^{[2]}$. This yields very explicit and relatively simple formulas. Note that this has also been worked out for principal 3-bundles with 2-crossed modules as structure 3-groups [GSTD25].

6.1. Example: principal bundles as principal 2-bundles

As a very instructive example, let us re-interpret an ordinary principal G-bundle as a principal $\mathcal{L}\mathsf{G}$-bundle, where $\mathcal{L}\mathsf{G}$ is again the crossed module weakly equivalent to G, trivially regarded as the crossed module $(* \to \mathsf{G})$.

The adjustment on $(* \to \mathsf{G})$ is evidently trivial. The adjustment on $\mathcal{L}\mathsf{G}$, however, is necessarily non-trivial, and of a very special form. Note that on $\mathcal{L}\mathsf{G}$, there is a map

$$P : P_0\mathfrak{g} \to L_0\mathfrak{g}\ , \quad (P\gamma)(\tau) = (\mathrm{id} - \tau \circ \flat)\gamma(\tau)\ , \tag{6.7}$$

such that

$$\partial \circ P \circ \partial = \partial \tag{6.8}$$

Proposition 6.2. *[RSW26] Given a map P satisfying (6.8), then*

$$\kappa_0(g,X) = P(gXg^{-1} - X) \tag{6.9}$$

satisfies adjustment relations up to terms in $\ker(\partial)$*.*

Exercise 6.1. Show this proposition for a general crossed module.

Note that $\ker(\partial) = *$ for $\mathcal{L}\mathsf{G}$, and therefore κ_0 is an adjustment on $\mathcal{L}\mathsf{G}$. Recall that we observed that the invariants of a crossed module are the kernel and the cokernel of ∂. On these, the adjustment data should be "non-trivial", while on the rest, it should be more or less canonical. The idea is that the adjustment κ_0 provides the "canonical" part.

We then have the following result:

Proposition 6.3. *[RSW26] The data in the differential cohomologies defining principal G-bundles with connection and principal $\mathcal{L}\mathsf{G}$-bundles with connection, both subordinate to some submersion $Y \to M$,*

$$g \in \Omega^0(Y^{[2]},\mathsf{G})\ , \quad A \in \Omega^1(Y,\mathfrak{g}) \tag{6.10}$$

and

$$\tilde{g} \in \Omega^0(Y^{[2]},P_0\mathsf{G})\ , \quad \tilde{h} \in \Omega^0(Y^{[3]},L_0\mathsf{G})\ , \quad \tilde{A} \in \Omega^1(Y,P_0\mathfrak{g})\ , \quad \tilde{B} \in \Omega^1(Y,L_0\mathfrak{g})\ , \tag{6.11}$$

respectively, is equivalent if we also impose

$$P(F) = 0\ . \tag{6.12}$$

This condition is called <u>adjusted fake flatness</u>.

6.2. Algebraic interpretation of infinitesimal adjustments

We can be much more general and still explicit, if we consider the "infinitesimal adjustments" that appear in the deformations required for the local BRST Lie-algebroid (i.e. the linearisation of the BRST Lie groupoid) of an L_∞-algebra-valued connection.

We already have a slight generalisation in the example of a 2-term L_∞-algebra $\mathfrak{L}$. Here, an adjustment is a map $\kappa : \mathfrak{L}_0 \times \mathfrak{L}_0 \to \mathfrak{L}_{-1}$ that satisfies

$$\kappa(\mu_1(Y), X) + \mu_2(X, Y) = 0 \ , \tag{6.13a}$$

$$\begin{aligned}\kappa(\mu_2(X_1, X_2), X_3) = \kappa\Big(X_1, \mu_2(X_2, X_3) - \mu_1(\kappa(X_2, X_3))\Big)&\\ + \mu_2(X_1, \kappa(X_2, X_3)) + \tfrac{1}{2}\mu_3(X_1, X_2, X_3) - (X_1 \leftrightarrow X_2) \ ,&\end{aligned} \tag{6.13b}$$

Even more concretely, we may consider the minimal string Lie 2-algebra $\mathbb{R} \xrightarrow{0} \mathfrak{spin}(n)$ from above, for which an adjustment is simply given by the Killing form

$$\kappa(X_1, X_2) \coloneqq (X_1, X_2) \ . \tag{6.14}$$

This is a general feature: if an adjustment exists, the required maps are usually evident and part of the data of the definition of the higher Lie group or algebra. The curvature is deformed as follows:

$$H = \mathrm{d}B - \tfrac{1}{3!}\mu_3(A, A, A) - \kappa(A, F) = \mathrm{d}B + \underbrace{\tfrac{1}{2}(A, \mathrm{d}A) + \tfrac{1}{3!}(A, [A, A])}_{\text{Chern–Simons 3-form}} \ , \tag{6.15}$$

and the adjustment completes the relatively strange expression $\frac{1}{3!}\mu_3(A, A, A)$ to the much more natural Chern–Simons 3-form. This particular form of adjusted gauge transformations and curvatures already appeared in physics papers in the context of "heterotic supergravity" in the 1980s [BdRdWvN82, CM83].

It is now very tempting to try to develop an algebraic or geometric interpretation of the adjustment map κ. A particularly nice algebraic interpretation has been obtained for these adjustments of L_∞-algebras [BKS21].

Note that in the categorification of a Lie algebra as a 2-term L_∞-algebra, we only lifted the Jacobi identity up to homotopy. One can, however, also lift the antisymmetry [Roy, BHR10]. We make the following definition.

Definition 6.4. [BKS21] An $h\mathcal{L}ie_2$-algebra is a graded vector space $\mathfrak{E} = \oplus_{k\in\mathbb{Z}}\mathfrak{E}_k$ together with a differential $\varepsilon_1 : \mathfrak{E} \to \mathfrak{E}$, which is a derivation of the map $\varepsilon_2 : \mathfrak{E} \times \mathfrak{E} \to \mathfrak{E}$ with $|\varepsilon_2| = 0$, as well as another map $\mathsf{alt} : \mathfrak{E} \times \mathfrak{E} \to \mathfrak{E}$ with $|\mathsf{alt}| = -1$. The map ε_2 is a Leibniz bracket and its antisymmetry is violated up to homotopy:

$$\varepsilon_2(x_1, x_2) - (-1)^{i+|x_1|\,|x_2|}\varepsilon_2(x_2, x_1) = \varepsilon_1(\mathsf{alt}(x_1, x_2)) + \big(\mathsf{alt}(\varepsilon_1(x_1), x_2) + (-1)^{|x_1|}\mathsf{alt}(x_1, \varepsilon_1(x_2))\big) \ , \tag{6.16}$$

and we also have

$$\begin{aligned}&\mathsf{alt}(\mathsf{alt}(x_1, x_2), x_3) = (-1)^{i(|x_1|+1)}\mathsf{alt}(x_1, \mathsf{alt}(x_2, x_3)) - (-1)^{(|x_1|+i)|x_2|}\mathsf{alt}(x_2, \mathsf{alt}(x_1, x_3)) \ ,\\ &\varepsilon_2(\mathsf{alt}(x_1, x_2), x_3) = 0 \ ,\\ &\mathsf{alt}(\varepsilon_2(x_1, x_2), x_3) = (-1)^{|x_1|}\varepsilon_2(x_1, \mathsf{alt}(x_2, x_3)) - (-1)^{|x_1|\,|x_2|}\mathsf{alt}(x_2, \varepsilon_2(x_1, x_3)) \ .\end{aligned} \tag{6.17}$$

There are now the following two propositions, which are a refinement of a result by Getzler [Get10] which, in turn, is a corollary to the results of [FM07].

Proposition 6.5. *[BKS21] Any $h\mathcal{L}ie_2$-algebra is weakly equivalent (in a category of homotopy $h\mathcal{L}ie_2$-algebras) to an L_∞-algebra by "anti-symmetrizing" the higher brackets.*

Proposition 6.6. *[BKS21] Every dgLa $(\mathfrak{g}, \mathsf{d}, [-,-])$ yields an $h\mathcal{L}ie_2$-algebra $(\mathfrak{E}, \varepsilon_1, \varepsilon_2, \mathsf{alt})$ by shifting and truncating the underlying complex:*

$$\mathfrak{E}_k = \begin{cases} \mathfrak{g}_{k-1} & k \leq 0 \\ * & \textit{else} \end{cases} \qquad\qquad \varepsilon_1(x) = \begin{cases} \mathsf{d}(x) & |x|_{\mathfrak{E}} < 0 \\ 0 & \textit{else} \end{cases} \tag{6.18}$$
$$\varepsilon_2(x_1, x_2) \coloneqq (\mathrm{d}x_1, x_2) \qquad\qquad \mathsf{alt}(x_1, x_2) \coloneqq [x_1, x_2]$$

A nice example for these statements from generalised geometry (that is, however, somewhat orthogonal to our discussion) is found in the Courant algebroid, a symplectic Lie 2-algebroid, where the differential graded algebra is given by the Poisson bracket and the Chevalley–Eilenberg differential. The $h\mathcal{L}ie_2$-bracket ε_2 is the Dorfman bracket, and the anti-symmetrized L_∞-bracket is the Courant bracket.

More important for us is the following result.

Proposition 6.7. *[BKS21] For any L_∞-algebra coming from an $h\mathcal{L}ie_2$-algebra via the above two propositions, the alternator provides an adjustment.*

As an example, consider the dgLa

$$\begin{gathered} \mathfrak{G} = \big(\mathfrak{G}_{-2} \xrightarrow{\mathsf{d}} \mathfrak{G}_{-1} \xrightarrow{\mathsf{d}} \mathfrak{G}_0\big) = \big(\mathbb{R} \xrightarrow{0} \mathfrak{spin}(n) \xrightarrow{\mathsf{id}} \mathfrak{spin}(n)\big) \\ [x_1, x_2]_{\mathfrak{G}} = [x_1, x_2] \ , \quad [x_1, y_1]_{\mathfrak{G}} = [x_1, y_1] \ , \quad [y_1, y_2]_{\mathfrak{G}} = (y_1, y_2) \end{gathered} \tag{6.19}$$

for all $x_i \in \mathfrak{G}_0$ and $y_i \in \mathfrak{G}_{-1}$ and all other brackets trivial. The shift-truncated $h\mathcal{L}ie_2$-algebra reads as

$$\mathfrak{E} = \big(\mathfrak{E}_{-1} \xrightarrow{\varepsilon_1} \mathfrak{E}_0\big) = \big(\mathbb{R} \xrightarrow{0} \mathfrak{spin}(n)\big) \ , \tag{6.20}$$

and the alternator is

$$\mathsf{alt}(y_1, y_2) = [y_1, y_2]_{\mathfrak{G}} = (y_1, y_2) \ . \tag{6.21}$$

The L_∞-algebra obtained by anti-symmetrisation is the minimal string Lie 2-algebra introduced above. The adjustment here comes indeed from the alternator:

$$\kappa(y_1, y_2) = (y_1, y_2), \tag{6.22}$$

and we fully recovered the adjusted form of the minimal string Lie 2-algebra.

We note that the connections in the kinematical data of all "gauged supergravity theories" are of a similar form. The initial data contains a dgLa, and our construction produces the corresponding adjusted bundle isomorphisms and curvatures.

7. Applications

Let us now go through the applications of our higher principal bundles with adjusted connections within mathematical physics. In particular, we will see some explicit examples of such bundles.

7.1. String bundles

Mathematically, string bundles are categorified notions of spin structures. In particular, a string structure on a manifold corresponds to a spin structure on the loop space of this manifold. Since fermions are described by sections of spinor bundles, i.e. associated vector bundles of spin bundles, this observation makes string structures also relevant to physics.

Moreover, we will see that special cases correspond to higher instantons, and instantons provided a vital bridge between mathematics and theoretical physics, that led to many interesting results.

String bundles can also be seen as trivialisations of the Chern–Simons 2-gerbe of a principal spin bundle P. This requires the first fractional Pontryagin class $\frac{1}{2}p_1$ to vanish. Beyond this, we shall ignore this perspective.

First, we recall a 2-group model of the string group [BSCS07], which is given by a particular central extension of $\mathcal{L}\mathsf{Spin}(n)$:

$$L_0\widehat{\mathsf{Spin}(n)} \xrightarrow{\partial} P_0\mathsf{Spin}(n) \ . \tag{7.1}$$

As shown in [RSW26], a suitable adjustment for this group is

$$\kappa(g,X) = \left(\kappa_0(g,X), \frac{\mathrm{i}}{2\pi}\int_0^1 \mathrm{d}\tau \left(g^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}g, X\right)\right) \ . \tag{7.2}$$

We can now construct an explicit principal 2-bundle with connection (see [RSW26] for all the details). Recall that the quaternionic Hopf fibration

$$\mathsf{Spin}(3) \cong S^3 \to S^7 \to S^4 \ , \tag{7.3}$$

where S^7 is the total space of a principal $\mathsf{Spin}(3)$-bundle over S^4. This example has $\frac{1}{2}\int p_1 = \frac{1}{2}$, so we cannot lift this bundle to a string structure.

However, we can double the instanton to an instanton-anti-instanton pair by considering the following fibration:

$$\mathsf{Spin}(3)^2 \cong \mathsf{Spin}(4) \to \mathsf{Spin}(5) \to S^4 \ , \tag{7.4}$$

where $\mathsf{Spin}(5)$ is now the total space of a principal $\mathsf{Spin}(4)$-bundle over S^4. Because of opposite signs in the Killing metric on the two $\mathsf{Spin}(3)$-factors in $\mathsf{Spin}(4)$, the Pontryagin classes in both factors cancel each other, and we can lift this bundle (topologically uniquely) to a principal $\mathsf{String}(4)$-bundle:

$$\mathsf{String}(4) \to \mathsf{String}(5) \to S^4 \ . \tag{7.5}$$

Note that indeed $\mathsf{String}(5)/\mathsf{String}(4) \cong S^4$. The idea to consider this principal 2-bundle is due to [Rob22], where also explicit Čech cocycles are given. The adjusted connection data is then found in [RSW26].

We stress that this principal 2-bundle with connection is completely explicit, both as a 2-group and in terms of differential cocycles. Also, the structure group $\mathsf{Spin}(3)^{\times 2} \cong \mathsf{SU}(2)^{\times 2}$ naturally appears in M-brane models, where one would expect categorified notions of instantons. Therefore, this principal 2-bundle is also of physical interest.

Finally we note that this bundle has $F \neq 0$, so that it is not "fake flat". Without adjustment, we could not possibly have described it.

7.2. Application: T-duality

The second big application of adjusted higher connections is within the principal 2-bundle interpretation of geometric T-duality. Recall that T-duality is essentially an equivalence of physical theories obtained by replacing the winding number of a physical string around an S^1-direction of spacetime with the discrete momentum number.

A simple string theory background suitable for T-duality consists of a principal circle bundle $\check{P}$ over some base manifold X and an abelian gerbe $\check{\mathscr{G}}$ (i.e. a principal $\mathsf{BU}(1)$-bundle) on the total space of $\check{P}$. We can now push down $\check{\mathscr{G}}$ by fibre integration, leading to a dual principal circle bundle $\hat{P}$. The existence of a Gysin sequence then ensures the existence of a dual gerbe $\hat{\mathscr{G}}$, whose push down corresponds to the principal bundle $\check{P}$, see [BEM04] for more details.

Alternatively, we can see this as a relation between the gerbes $\check{\mathscr{G}}$ and $\hat{\mathscr{G}}$, pulled back over the correspondence space $\check{P} \times_X \hat{P}$ [BS05, BRS06]:

$$
\begin{tikzcd}
 & & \mathscr{G}_{\mathrm{C}} = \check{\mathsf{p}}^*\check{\mathscr{G}} \otimes \hat{\mathsf{p}}^*\hat{\mathscr{G}}^{-1} \xrightarrow[\cong]{\mathcal{P}_\nabla} \mathbb{I} \arrow[d] & & \\
 & & \check{P} \times_X \hat{P} \arrow[dl, "\check{\mathsf{p}}"'] \arrow[dr, "\hat{\mathsf{p}}"] & & \\
\check{\mathscr{G}} \arrow[r] & \check{P} \arrow[dr, "\check{\pi}"] & & \hat{P} \arrow[dl, "\hat{\pi}"'] & \hat{\mathscr{G}} \arrow[l] \\
 & & X & &
\end{tikzcd}
\tag{7.6}
$$

Here, the pullback of $\check{\mathscr{G}}$ tensored by the inverse gerbe of the pullback of $\hat{\mathscr{G}}$ is trivial, and there is a trivialisation that is given by the Poincaré line bundle $\mathcal{P}_\nabla$ over the correspondence space.

This picture was then re-interpreted in terms of spans of principal 2-bundles in [NW19]. In particular, a string background, i.e. an abelian gerbe over a principal n-torus bundle, was turned into a principal TB_n-bundle, where TB_n is a 2-group whose exact structure is irrelevant. One can show that the classifying space of TB_n is homotopy equivalent to the classifying space BTD_n of the following 2-group, given as a crossed module:

$$
\begin{aligned}
\mathsf{TD}_n \;&:=\; \left(\mathbb{Z}^{2n} \times \mathsf{U}(1) \xrightarrow{\mathsf{t}} \mathbb{R}^{2n}\right) , \\
\mathsf{t}(m, \phi) &:= m , \\
\xi \triangleright (m, \phi) &:= (m, \phi - \langle \xi, m \rangle) ,
\end{aligned}
\tag{7.7a}
$$

where

$$
\langle \xi_1, \xi_2 \rangle = \xi_1^{\mathrm{T}} \begin{pmatrix} 0 & 0 \\ \mathbb{1}_n & 0 \end{pmatrix} \xi_2 . \tag{7.7b}
$$

This 2-group describes a resolved $2n$-dimensional torus ($\mathbb{Z}^{2n} \hookrightarrow \mathbb{R}^{2n}$) together with a $\mathsf{BU}(1)$-factor describing the abelian gerbe.

Because of the homotopy equivalence $\mathsf{BTD}_n \cong \mathsf{BTB}_n$, every string background or principal TB_n-bundle is obtained from a principal TD_n-bundle. Moreover, TD_n contains a natural action of the T-duality group $\mathsf{O}(n, n; \mathbb{Z})$, which contains the flip transformation F corresponding to the T-duality. Altogether, we have a span of 2-groups inducing a span of principal 2-bundles:

$$
\begin{tikzcd}
 & \mathsf{TD}_n \arrow[dl, "\check{\mathsf{p}}"'] \arrow[dr, "\hat{\mathsf{p}}"] & \\
\mathsf{TB}_n & & \mathsf{TB}_n
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathscr{P}_{\mathrm{C}} \arrow[dl, "\check{\mathsf{p}}"'] \arrow[dr, "\hat{\mathsf{p}}"] & \\
\check{\mathscr{P}} & & \hat{\mathscr{P}}
\end{tikzcd}
\tag{7.8}
$$

where

$$
\hat{\mathsf{p}} = \check{\mathsf{p}} \circ F . \tag{7.9}
$$

For full geometric T-duality (at least for the so-called NS-NS-sector), we also need to endow the bundle $\mathscr{P}_{\mathrm{C}}$ with a connection, and here, it is again unavoidable to work with adjusted connections [KS22, KS23]. As shown there, an adjustment for TD_n is given by the natural map

$$
\kappa \colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{Z}^{2n} \times \mathsf{U}(1) , \quad (\xi; X) \mapsto (0, -\langle X, \xi \rangle) . \tag{7.10}
$$

Interestingly, one can show that there is no adjustment for TB_n.

Exercise 7.1. Show this theorem. (Definition of TB_n is in [KS22].)

7.3. Lie groupoid bundles

Let us also briefly mention the notion of adjustments for Lie groupoid bundles as found in [FJFKS24], since these provide a particularly nice interpretation of the adjustment datum. Also, this class of examples arises in ordinary physics, without invoking the notion of any higher structures.

Consider the Standard Model of Particle Physics, which is an example of a gauge–matter field theory. In such a classical field theory, the kinematic data consists of two parts: the "gauge part" is a connection on a principal fibre bundle, and the "matter part" is a section of an associated vector bundle. An alternative point of view is obtained by combining the group G and its relevant representation V into an action groupoid $\mathscr{G} = (V \rtimes \mathsf{G} \rightrightarrows V)$. Then the kinematic data is simply a principal $\mathscr{G}$-bundle.

For $\mathscr{G}$ a groupoid, a principal $\mathscr{G}$-bundle over a manifold M and subordinate to a surjective submersion $\sigma : Y \to M$ is given by a functor

$$(M \rightrightarrows M) \xrightarrow{\cong} \check{\mathscr{C}}(\sigma) = (Y^{[2]} \rightrightarrows Y) \xrightarrow{(\phi, A)} \mathscr{G} \ . \tag{7.11}$$

Just as in the case of higher connections, we also have now here curvatures of two form degrees: the covariant derivative of the matter field ϕ, which is a 1-form, and the usual curvature 2-form of A. Therefore, again, we usually have to define an adjustment κ.

As shown in [FJFKS24], such an adjustment amounts to defining a Cartan connection on the groupoid $\mathscr{G}$, which is a particularly nice interpretation of the adjustment datum.

We note that for action groupoids, this Cartan connection can be trivial, and therefore we usually don't see adjustment data in physical field theories. However, this does not mean that it has to be trivial, and non-trivial Cartan connections may lead to physically interesting deformations of the standard model of elementary particle physics.

7.4. Physical theories containing higher gauge theories

Recall that in physics, we usually describe a classical field theory by an "action functional" $S[\Phi]$, a functional on the "field space" given by the kinematical data. The classical equations of motion (i.e. differential equations (on local data) determining the dynamics) are then obtained as the stationary points of the action functional: $\frac{\delta S[\Phi]}{\delta \Phi} = 0$.

There are now a number of classical field theories involving connections on a higher principal bundle:

- We have already encountered heterotic and gauged supergravity theories above. Here, connections on principal 2-bundles are important in making the theory independent of the spacetime manifold topology.
- One of the "holy grails" in string theory is a particular superconformal field theory in 6d, usually called "the (2,0)-theory" or, more dramatically, "theory X". The existence of this field theory was predicted three decades ago [Wit95], and there are a number of no-go arguments that say that this theory is a pure quantum theory without a classical description. A good part of these no-go arguments can be refuted by using adjusted connections on higher principal bundles, but there are serious arguments remaining, which most likely mean that there is no classical description of this theory.
- One can, however, develop an explicit description of a slightly less symmetric theory, the $(1, 0)$-superconformal field theories in six dimensions, see [RSvdW21] and references therein.

But there are many further examples.

We note that a further higher contender would be higher Chern–Simons theory (with plenty of mathematical interest, e.g. for a potential description of higher knot invariants). Recall that, operadically, "$Com \otimes Lie = Lie$". This generalises at the homotopy algebraic level to the fact that a differential graded commutative algebra tensored by an L_∞-algebra is naturally an L_∞-algebra.

Consider then the tensor product L_∞-algebra

$$\hat{\mathfrak{L}} = \Omega^\bullet(M) \otimes \mathfrak{L} , \qquad (7.12)$$

where $\mathfrak{L}$ is a cyclic[13] n-term L_∞-algebra and M is a $2+n$-dimensional manifold. The homotopy Maurer-Cartan theory on $\hat{\mathfrak{L}}$ then is a natural form of higher Chern–Simons theory [JRSW19].

Although the equations of motion of Chern–Simons theory are equivalent to flatness of the connection, we would still expect the quantum theory to involve non-flat connections, so that we would like to work with adjusted connections. Unfortunately, one can show [GRSW25] that cyclic minimal n-term L_∞-algebras with adjustment are abelian, i.e. $\mu_n = 0$ for $n \geq 2$. Since higher Chern–Simons theory respects quasi-isomorphisms of gauge L_∞-algebras, this implies that there is no interesting form of higher Chern–Simons theory, and the situation remains unsatisfactory. It may be that higher gauge theory splits into physical theories, requiring adjustments, and topological theories, requiring other generalised forms of higher connections.

8. Conclusion

In these lecture notes, we developed the notion of adjusted connections on higher principal bundles, and presented a number of examples. The definition of higher connections proved to be a bit subtle and does not straightforwardly follow from vertical categorification. The definition presented here, however, is mathematically fully consistent. Moreover, these are the connections that we know from a number of physical field theories that naturally appear in the context of string and M-theory.

Since these lectures are presented to a large audience of young mathematicians, let me also list a number of open questions that I would very much like to see answers to.

- What are adjustments generally, geometrically? We saw above some initial interpretation of adjustment data in terms of alternators of higher Lie algebras and connections on Lie groupoids. Is there a unifying picture?
- When do adjustments exist? When are they unique? We have seen above that there are 2-groups, such as TB_n, for which we do not have adjustments. Can we always replace these with other 2-groups that have homotopy-equivalent classifying spaces, for which adjustments then exist?
- What is the best form for higher parallel transport? There is an initial notion contained in [KS20], but I feel that one can improve on this.
- What is the right notion of higher Chern–Simons theory? As explained in the previous section, the definition of higher Chern–Simons theory may require other generalised forms of higher connections.
- Finally, there is a big question that I currently do not understand at all: Many physical field theories, such as the heterotic and gauged supergravity theories described above do not respect weak equivalences of structure groups. This seems mathematically completely unnatural, yet the physics is very clear that these are the preferred theories. Why is this?

[13] A cyclic structure on an n-term L_∞-algebra is the appropriate notion of invariant inner product.

Appendix

A. Hints for solving the exercises

Below are some hints on how to solve the exercises or even references to the full solutions.

Exercise 2.1: put some of the g_i/g'_i to unit to make the two required arguments work.

Exercise 2.2: Note that many assumptions do not work. Source and target 1-morphisms generally won't match. Also, horizontal/vertical units do not coincide.

Exercise 3.1: Assume a butterfly between two crossed modules $(\mathsf{H}_{1,2} \xrightarrow{\partial_{1,2}} \mathsf{G}_{1,2})$ and show that $\ker(\partial_1) \cong \ker(\partial_2)$ and $\mathrm{coker}(\partial_1) \cong \mathrm{coker}(\partial_2)$. The first is done by diagram chasing: pick an element $Y_1 \in \ker(\partial_1)$, map it to E via λ_1. Then there is a unique element $Y_2 \in \mathsf{H}_2$ that maps to the same element in E via λ_2. One can then show that this element is in $\ker(\partial_2)$.

For coker(...) express it as a quotient of E by the subgroup generated by both $\mathrm{im}(\lambda_1)$ and $\mathrm{im}(\lambda_2)$.

Exercise 3.2: This is a straightforward exercise. The action of $P_0\mathsf{G}$ onto $L_0\mathsf{G}$ is the pointwise adjoint action.

Exercise 3.3: Look at [JSW15, Section 3]. This treats the case of weak 2-groups and then lists the restriction to strict 2-groups.

Exercise 4.1: The underlying complexes are the same. For the Lie bracket μ_2 on $\mathfrak{G}$, we have

$$[X_1, X_2]_{\mathfrak{g}} = \mu_2(X_1, X_2) \quad X \triangleright Y = \mu_2(X, Y) \quad [Y_1, Y_2] = \mu_2(\mu_1(Y_1), Y_2) \tag{A.1}$$

for $X, X_{1,2} \in \mathfrak{g}$ and $Y, Y_{1,2} \in \mathfrak{h}$. This is due to the linearisation of the second axiom for crossed modules of groups.

Exercise 4.2: The Chevalley–Eilenberg algebra of a Lie 2-algebra is explained e.g. in [SS20, Section 2.1], where you find also a useful trick for keeping track of the various signs.

Exercise 4.3: Choose a sign convention, e.g. the one from the previous exercise, and consider a map Φ from the generators of one CE-algebra to all possible generators and products of generators in another one. Up to signs, you should be able to reproduce the general formula in [JRSW19, Section 2.4].

Exercise 4.4: The explicit form of the quasi-isomorphism in both directions is found e.g. in the discussion of the minimal model of the string Lie 2-algebra in [RSW26, Section 4.1].

Exercise 6.1: This is the proof of [RSW26, Prop. 2.7].

Exercise 7.1: This is a simplification of [KS22, Theorem 4.4].

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